

b $P(n) : un = n^2 - 2n + 2$

$$P(1) \Rightarrow u_1 = 1^2 - 2(1) + 2 = 1 \quad \therefore P(1) \text{ is true}$$

$$\text{Assume } P(k) \quad u_k = k^2 - 2k + 2$$

$$\begin{aligned} \text{Prove } P(k+1) \quad u_{k+1} &= u_k + 2(k+1) - 3 \\ &= k^2 - 2k + 2 + 2k + 2 - 3 \\ &= k^2 + 1 \end{aligned}$$

$$\begin{aligned} (k+1)^2 - 2(k+1) + 2 &= k^2 + 2k + 1 \\ &\quad - 2k - 2 + 2 \\ &= k^2 + 1 \end{aligned}$$

$$\therefore u_{k+1} = (k+1)^2 - 2(k+1) + 2$$

$\therefore P(k) \Rightarrow P(k+1)$ and $P(1)$ is true

\therefore by mathematical induction, $u_n = n^2 - 2n + 2$

Exercise 5B

1 a $(64)^{\frac{2}{3}} = 4^2 = 16$

b $\left(\frac{8}{27}\right)^{\frac{1}{3}} = \frac{2}{3}$

c $\left(\frac{81}{16}\right)^{-\frac{3}{4}} = \left(\frac{16}{81}\right)^{\frac{3}{4}} = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

2 a $\left(\frac{b^{-3}x^{-2}}{8x}\right)^{\frac{-2}{3}} = \left(\frac{8x}{b^{-3}x^{-2}}\right)^{\frac{2}{3}} = (8x^3b^3)^{\frac{2}{3}}$

$$\therefore \left(\frac{b^{-3}x^{-2}}{8x}\right)^{\frac{-2}{3}} = (2xb)^2 = 4b^2x^2$$

b $\frac{a^{-1}-a^{-2}}{a^{-3}} = \frac{a^{-1}}{a^{-3}} - \frac{a^{-2}}{a^{-3}} = a^2 - a = a(a-1)$

c $\frac{x^3 \times x^{-7}}{x^{-4}} = \frac{x^{-4}}{x^{-4}} = 1$

3 $\sqrt{y^3} \div \sqrt[3]{y^2} = y^{\frac{3}{2}} \div y^{\frac{2}{3}} = y^{\left(\frac{3}{2}-\frac{2}{3}\right)} = y^{\frac{5}{6}}$

when $y = 64$, $y^{\frac{5}{6}} = (\sqrt[6]{64})^5 = 2^5 = 32$

4
$$\begin{aligned} \frac{(x^4yz^{-3})^2 \times \sqrt{x^{-5}y^2z}}{(xz)^{\frac{7}{2}}} &= \frac{x^8y^2z^{-6} \times x^{-\frac{5}{2}}yz^{\frac{1}{2}}}{x^{\frac{7}{2}}z^{\frac{7}{2}}} \\ &= \frac{x^{\frac{11}{2}}y^3z^{-\frac{11}{2}}}{x^{\frac{7}{2}}z^{\frac{7}{2}}} = x^2y^3z^{-9} \\ &= \frac{x^2y^3}{z^9} \end{aligned}$$

5
$$\begin{aligned} 5 \times 4^{3n+1} - 20 \times 8^{2n} &= 5 \times (2^2)^{3n+1} - 20 \times (2^3)^{2n} \\ &= 5 \times 2^{6n+2} - 20 \times 2^{6n} \\ &= 5 \times 2^2 \times 2^{6n} - 20 \times 2^{6n} \\ &= 20 \times 2^{6n} - 20 \times 2^{6n} \\ &= 0 \end{aligned}$$

6 $4^x + 2 = 3 \times 2^x$

$$(2^x)^2 - 3(2^x) + 2 = 0$$

$$(2^x - 1)(2^x - 2) = 0$$

$$2^x = 1 \text{ or } 2^x = 2$$

$$x = 0 \text{ or } 1$$

Exercise 5C

1 $250000(1+r)^{20} = 450000$

$$(1+r)^{20} = 1.8$$

$$1+r = 1.0298$$

$$r = 0.0298 = 2.98\% = 3\% \text{ (nearest percent)}$$

2 a $61.08 = 17.48(1+r)^7$

$$(1+r)^7 = 3.494$$

$$1+r = 1.1957$$

$$r = 0.196 = 19.6\%$$

b $77.45 = 61.08(1+r)^4$

$$(1+r)^4 = 1.268\dots$$

$$1+r = 1.0612$$

$$r = 0.0612 = 6.12\%$$

c $97.87 = 72.99(1+r)^{12}$

$$(1+r)^{12} = 1.3408$$

$$1+r = 1.0247$$

$$r = 0.0247 = 2.47\%$$

3 Samira: After 15 years (60 quarters), she will have

$$\begin{aligned} 1000 \left(1 + \frac{0.08}{4}\right)^{60} &= 1000 \times (1.02)^{60} \\ &= \$3280 \end{aligned}$$

$$\begin{aligned} \text{Hemanth: } 500(1.08)^{15} + 500 \left(1 + \frac{0.084}{12}\right)^{12 \times 15} &= 500(1.08)^{15} + 500(1.007)^{180} \\ &= 1586.08 + 1754.99 \\ &= \$3340 \end{aligned}$$

End of year	Amount owing	Pays back
1	$15000 \times 1.05 = 15750$	10500
2	$5250 \times 1.05 = 5512.5$	3675
3	$1837.5 \times 1.05 = 1929.375$	1286.25
4	$643.125 \times 1.05 = 675.28$	450.1875
5	$225.09375 \times 1.05 = 236.348$	157.565625
		16069.00313

\therefore Guiseppe has paid back approx. € 16700

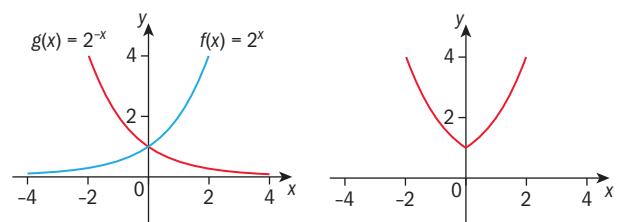
Exercise 5D

1 Red curve: $(1, 2.5) \quad 2.5 = a^1 \quad \therefore a = 2.5$

Blue curve: $(-1, 4) \quad 4 = a^{-1} \quad \therefore a = \frac{1}{4}$

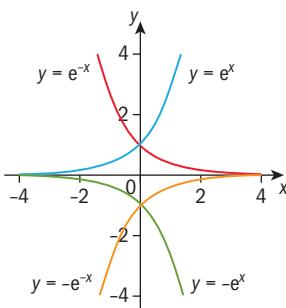
2 $f(x) = 2^x \quad f(x+1) = 2^{(x+1)} = 2(2^x) = 2f(x)$
 $f(x+a) = 2^{x+a} = 2^a(2^x) = 2^a f(x)$

3



4 $e^x + 1 = e^{x+1}$
 $x = -0.541$

- 5 a reflection in the y -axis
 b reflection in the x -axis
 c rotation of 180° about the origin (or reflection in the y -axis followed by reflection in the x -axis)



Exercise 5E

- 1 a $5^3 = 125 \Rightarrow \log_5 125 = 3$
 b $10^3 = 1000 \Rightarrow \log_{10} 1000 = 3$
 c $27^{\frac{1}{3}} = 3 \Rightarrow \log_{27} 3 = \frac{1}{3}$
 d $10^{-3} = 0.001 \Rightarrow \log_{10} 0.001 = -3$
 e $m = n^2 \Rightarrow \log_n m = 2$
 f $a^b = 2 \Rightarrow \log_a 2 = b$
- 2 a $\log_3 9 = 2 \Rightarrow 3^2 = 9$
 b $\log_{10} 1000000 = 6 \Rightarrow 10^6 = 1000000$
 c $\log_{49} 7 = \frac{1}{2} \Rightarrow 49^{\frac{1}{2}} = 7$
 d $\log_a 1 = 0 \Rightarrow a^0 = 1$
 e $\log_4 a = b \Rightarrow 4^b = a$
 f $\log_p q = r \Rightarrow p^r = q$
- 3 a $\log_8 64 = 2$
 b $\log_9 3 = \frac{1}{2}$
 c $\log_{10} 0.01 = -2$
 d $\log_{144} 12 = \frac{1}{2}$
 e $\log_{37} 1 = 0$
 f $\log_a \sqrt[3]{a} = \frac{1}{3}$
- 4 a $\log_x 81 = 2 \therefore x^2 = 81 \therefore x = 9$
 b $\log_3 x = 4 \therefore 3^4 = x \therefore x = 81$
 c $\log_{11} 121 = x \therefore x = 2$
 d $\log_x 5 = \frac{1}{3} \therefore x^{\frac{1}{3}} = 5 \therefore x = 125$
 e $\log_x 16 = \frac{2}{3} \therefore x^{\frac{2}{3}} = 16 \therefore x = 16^{\frac{3}{2}} = 4^3 = 64$
 f $\log_x 32 = -5 \therefore x^{-5} = 32 \therefore \frac{1}{x^5} = 32, x^5 = \frac{1}{32}, x = \frac{1}{2}$

Exercise 5F

- 1 a $\log_a \frac{p^2}{q} = 2 \log_a p - \log_a q$
 b $\log_a \sqrt[3]{\frac{p}{q^2}} = \frac{1}{3} \log_a \frac{p}{q^2} = \frac{1}{3} \log_a p - \frac{2}{3} \log_a q$

- 2 a $\log 4 + 2 \log 3 - \log 6 = \log \frac{4 \times 9}{6} = \log 6$
 b $\frac{1}{2} \log_a p + \frac{1}{4} \log_a q^2 = \log_a p^{\frac{1}{2}} + \log_a q^{\frac{1}{2}} = \log_a \sqrt{pq}$
 c $2 - \log 5 = \log 100 - \log 5 = \log 20$
 3 a $\log 5 + \log 8 - \log 4 = \log \frac{5 \times 8}{4} = \log 10 = 1$
 b $\log_2 48 - \frac{1}{3} \log_2 27 = \log_2 48 - \log_2 3 = \log_2 16 = 4$
 c $2 + \log_5 10 - \log_5 2 = 2 + \log_5 5 = 2 + 1 = 3$
- 4 a $3 \log y = 2 \log x \Rightarrow y^3 = x^2 \therefore y = x^{\frac{2}{3}}$
 b $\log y = \log x + \log 2 \Rightarrow y = 2x$
 c $\log y - 3 \log x = \log 2 \Rightarrow \frac{y}{x^3} = 2 \therefore y = 2x^3$
 d $\log y = 2 + 3x \Rightarrow y = 10^{2+3x}$

Exercise 5G

- 1 a $\log_3 2 \times \log_3 81 = \log_3 2 \times \frac{\log_3 81}{\log_3 2} = 4$
 b $\log_6 10 \times \log 6 = \log_6 10 \times \frac{\log_6 6}{\log_6 10} = 1$
 c $\log_{125} 8 \times \log_8 5 = \log_{125} 8 \times \frac{\log_8 5}{\log_{125} 8} = \frac{1}{3}$
 d $\frac{1}{\log_2 6} + \frac{1}{\log_3 6} = \frac{1}{\log_2 6} + \frac{\log_2 3}{\log_2 6}$
 $= \frac{\log_2 2 + \log_2 3}{\log_2 6} = \frac{\log_2 6}{\log_2 6} = 1$
 e $\frac{1}{\log_4 6} + \frac{1}{\log_9 6} = \frac{1}{\log_4 6} + \frac{\log_4 9}{\log_4 6} = \frac{\log_4 4 + \log_4 9}{\log_4 6}$
 $= \frac{\log_4 36}{\log_4 6} = \frac{2 \log_4 6}{\log_4 6} = 2$
 f $\log_5 40 - \frac{1}{\log_8 5} = \log_5 40 - \frac{\log_5 8}{\log_5 5} = \log_5 5 = 1$
- 2 a let $a^{\log b} = x \Rightarrow \log_a x = \log b$
 $\frac{\log x}{\log a} = \log b$
 $\frac{\log x}{\log b} = \log a$
 $\log_b x = \log a$
 $\therefore x = b^{\log a}$
 $\therefore a^{\log b} = b^{\log a}$
- b $\frac{1}{\log_a ab} + \frac{1}{\log_b ab} = \frac{\log_{ab} a}{\log_{ab} ab} + \frac{\log_{ab} b}{\log_{ab} ab}$
 $= \log ab^a + \log ab^b$
 $= \log ab^{(ab)}$
 $= 1$
 $\therefore \frac{1}{\log_a ab} + \frac{1}{\log_b ab} = 1$
- 3 $p = \log a^x \quad q = \log a^y$
 $\log_x a = \frac{1}{\log_a x} = \frac{1}{p} \quad \log_y a = \frac{1}{\log_a y} = \frac{1}{q}$
 a $\log_{xy} a = \frac{\log_a a}{\log_a xy} = \frac{1}{\log_a x + \log_a y} = \frac{1}{p+q}$
 b $\log_{xy} a = \frac{1}{\log_a \left(\frac{x}{y}\right)} = \frac{1}{\log_a x - \log_a y} = \frac{1}{p-q}$

Exercise 5H

1 a $5^x = 7$
 $x \log 5 = \log 7$
 $x = \frac{\log 7}{\log 5}$
 $x = 1.21$

b $4^{2x-1} = 3$
 $(2x - 1) \log 4 = \log 3$
 $2x - 1 = \frac{\log 3}{\log 4}$
 $x = \frac{1}{2} \left(\frac{\log 3}{\log 4} + 1 \right)$
 $\therefore x = 0.896$

2 $(2^x)(5^x) = 0.01$
 $10^x = 0.01$
 $\therefore x = -2$

Exercise 5I

1 a $2^{3x} = 5$
 $3x \log 2 = \log 5$
 $x = \frac{\log 5}{3 \log 2}$
 $x = 0.774$

b $3^x (3^{x-1}) = 10$
 $3^{2x-1} = 10$
 $(2x - 1) \log 3 = \log 10$
 $2x - 1 = \frac{1}{\log 3}$
 $x = \frac{1}{2} \left(\frac{1}{\log 3} + 1 \right)$
 $x = 1.55$

2 a $4 \log_3 x = \log_x 3$
 $4 \log_3 x = \frac{1}{\log_3 x}$
 $(\log_3 x)^2 = \frac{1}{4} \quad \therefore \log_3 x = \pm \frac{1}{2}$
 $x = 3^{\frac{1}{2}} \text{ or } x = 3^{-\frac{1}{2}}$

b $3 \log_2 x + \log_2 27 = 3$
 $\log_2 (27x^3) = 3$
 $\therefore 27x^3 = 8$
 $x^3 = \frac{8}{27}$
 $x = \frac{2}{3}$

3 $9^x - 6(3^x) - 16 = 0$
 $(3^x)^2 - 6(3^x) - 16 = 0$
 $(3^x - 8)(3^x + 2) = 0$
 $3^x = 8$
 $x \log 3 = \log 8, x = 1.89$

4 $\log_4 x + 12 \log_x 4 - 7 = 0$
 $\log_4 x + \frac{12}{\log_4 x} - 7 = 0$
 $(\log_4 x)^2 - 7 \log_4 x + 12 = 0$
 $(\log_4 x - 3)(\log_4 x - 4) = 0$
 $\log_4 x = 3 \text{ or } \log_4 x = 4$
 $x = 4^3 \text{ or } x = 4^4$
 $x = 64 \text{ or } 256$

5 $5^{x+1} + \frac{4}{5^x} - 21 = 0$
 $5(5^x)^2 - 21(5^x) + 4 = 0$
 $(5(5^x) - 1)(5^x - 4) = 0$
 $5^x = \frac{1}{5} \quad \text{or} \quad 5^x = 4$
 $x = -1 \quad \text{or} \quad x \log 5 = \log 4$
 $x = -1 \quad \text{or} \quad 0.861$

6 $\log_3 x + \log_x 9 - 3 = 0$
 $\log_3 x + \frac{\log 9}{\log_3 x} - 3 = 0$
 $(\log_3 x)^2 - 3 \log_3 x - 2 = 0$
 $(\log_3 x - 2)(\log_3 x - 1) = 0$
 $\log_3 x = 2 \quad \text{or} \quad \log_3 x = 1$
 $x = 3^2 \quad \text{or} \quad x = 3^1$
 $x = 3 \quad \text{or} \quad 9$
7 $3 \times 9^x - 2 \times 4^x = 5 \times 6^x$
 $3(3^x)^2 - 5(2^x)(3^x) - 2(2^x)^2 = 0$
 $(3(3^x) + 2^x)(3^x - 2(2^x)) = 0$
 $3(3^x) = -2^x \quad \text{or} \quad 3^x = 2(2^x)$
 $3(1.5)^x = -1 \quad \text{or} \quad 1.5^x = 2$
 No solution, $x \log 1.5 = \log 2$
 $x = 1.71$

8 $6 \log_2 x + 6 \log_2 y = 7$
 $6 \log_2 x + \frac{6 \log_2 y}{\log_2 8} = 7$
 $6 \log_2 x + 2 \log_2 y = 7$
 $\log_2 x^6 y^2 = 7$
 $x^6 y^2 = 2^7 \quad \therefore x^6 y^2 = 128 \quad (1)$

$$\begin{aligned} 4 \log_4 x + 4 \log_2 y &= 9 \\ \frac{4 \log_2 x}{\log_2 4} + 4 \log_2 y &= 9 \\ 2 \log_2 x + 4 \log_2 y &= 9 \\ \log_2 x^2 y^4 &= 9 \\ \therefore x^2 y^4 &= 2^9 \quad \therefore x^2 y^2 = 512 \quad (2) \end{aligned}$$

$$\begin{aligned} \text{from (1)} \quad y^2 &= \frac{128}{x^6} \quad x^2 \left(\frac{128}{x^6} \right)^2 = 512 \\ 16384 &= 512x^{10} \quad \therefore x^{10} = 32 \\ y^2 &= \frac{128}{8} = 16 \quad x = \sqrt[10]{32} \\ y &= 4 \end{aligned}$$

9 $2\log xy = 1 \Rightarrow x = y^2$

$$xy = 125 \quad \therefore y^3 = 125$$

$$y = 5, x = 25$$

10 $y \log_2 8 = x \Rightarrow 3y = x$

$$2^x + 8^y = 64$$

$$2^{3y} + 2^{3y} = 64$$

$$2^{3y+1} = 64$$

$$3y + 1 = 6$$

$$y = \frac{5}{3}, \quad x = 5$$

11 a $\log_5 x = y = \log_{25}(2x - 1)$

$$x = 5^y \quad 25^y = 2x - 1$$

$$(5^y)^2 = 2x - 1$$

$$x^2 - 2x + 1 = 0$$

$$(x-1)^2 = 0$$

$$x = 1, y = 0$$

b $\log(x + y) = 0 \Rightarrow x + y = 1 \Rightarrow y = 1 - x$

$$2\log x = \log(y + 5) \Rightarrow x^2 = y + 5$$

$$x^2 = 1 - x + 5$$

$$x^2 + x - 6 = 0$$

$$(x+3)(x-2) = 0$$

$$x = 2 \quad y = -1$$

(x cannot be negative)

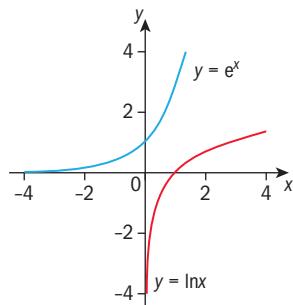
Exercise 5J

1 $f(x) = e^x$: domain is $x \in \mathbb{R}$

range is $y \in \mathbb{R}, y > 0$

$$f^{-1}(x) = \ln x$$
: domain is $x \in \mathbb{R}, x > 0$

range is $y \in \mathbb{R}$



2 $f(x) = a^x \quad y = a^x$

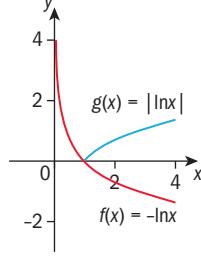
Inverse: $x = a^y, \quad y = \log_a x$

$$\therefore f^{-1}(x) = \log_a x$$

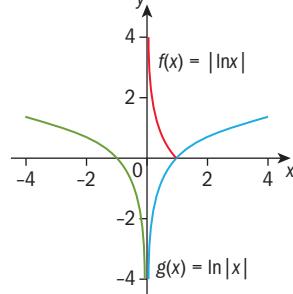
$$f \circ f^{-1}(x) = f(\log_a x) = a^{\log_a x}$$

$$\therefore a^{\log_a x} = x$$

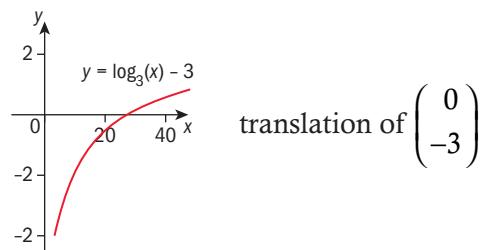
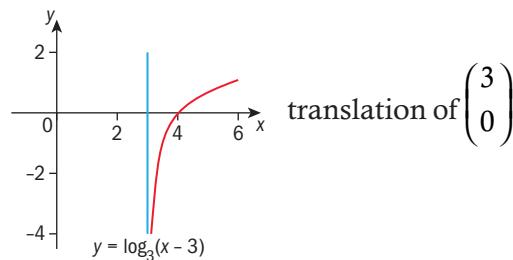
3



4



5



6 a $y = \ln(x-1) - 1$

$$x-1 > 0 \quad \therefore x > 1$$

domain: $x \in \mathbb{R}, x > 1$

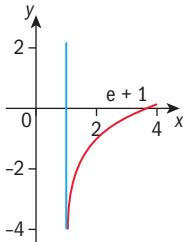
asymptote: $x = 1$

$$x\text{-intercept } 0 = \ln(x-1) - 1$$

$$\ln(x-1) = 1$$

$$(e+1, 0) \quad x-1 = e$$

$$x = e + 1$$



b $y = \log_3(9-3x) + 2$

$$9-3x > 0$$

$$9 > 3x$$

$$x < 3$$

domain: $x \in \mathbb{R}, x < 3$

asymptote: $x = 3$

$$x\text{-intercept } 0 = \log_3(9-3x) + 2$$

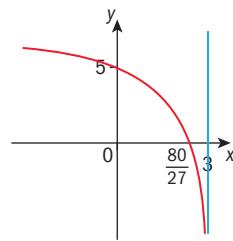
$$-2 = \log_3(9-3x)$$

$$3^{-2} = 9-3x$$

$$3x = 9 - \frac{1}{9} = \frac{80}{9}$$

$$x = \frac{80}{27} \quad \left(\frac{80}{27}, 0 \right)$$

y -intercept $y = \log_3 9 + 2 = 4(0, 4)$



Exercise 5K

1 a $y = \frac{3}{2} e^{x^2} \frac{d^2y}{dx^2} = \frac{3}{2} (2xe^{x^2}) = 3xe^{x^2}$

b $y = \frac{-5}{e^{3x-1}} = -5e^{-(3x-1)} \quad \frac{dy}{dx} = 15e^{-(3x-1)} = \frac{15}{e^{3x-1}}$

c $y = e^{4x-1} + 4 \quad \frac{dy}{dx} = 4e^{4x-1}$

d $y = e^x + \frac{1}{e^x} = e^x + e^{-x} \frac{dy}{dx} = e^x - e^{-x} = e^x - \frac{1}{e^x}$

e $y = e^{-(1-3x)} \frac{dy}{dx} = 3e^{-(1-3x)}$

f $y = 2e^{\sqrt{x}} \frac{dy}{dx} = 2\left(\frac{1}{2}x^{-\frac{1}{2}}\right)e^{\sqrt{x}} = \frac{e^{\sqrt{x}}}{\sqrt{x}}$

2 a $y = xe^x \quad \frac{dy}{dx} = e^x + xe^x$

b $y = \frac{x^2}{e^x} = x^2e^{-x} \quad \frac{dy}{dx} = 2xe^{-x} - x^2e^{-x} = \frac{2x-x^2}{e^x}$

c $y = \frac{e^{2x}}{\sqrt{x}} = e^{2x}x^{-\frac{1}{2}} \quad \frac{dy}{dx} = 2e^{2x}x^{-\frac{1}{2}} - e^{2x}\frac{1}{2}x^{-\frac{3}{2}}$
 $= \frac{2e^{2x}}{x^{\frac{1}{2}}} - \frac{e^{2x}}{2x^{\frac{3}{2}}} = \frac{4xe^{2x} - e^{2x}}{2x^{\frac{3}{2}}}$

d $y = \sqrt{x}e^{\sqrt{x}} \quad \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}e^{\sqrt{x}} + \sqrt{x}\frac{1}{2}x^{-\frac{1}{2}}e^{\sqrt{x}}$
 $= \frac{e^{\sqrt{x}}}{2\sqrt{x}} + \frac{1}{2}e^{\sqrt{x}}$

3 a $y = \frac{e^{2x}}{\sqrt{x}} \quad \frac{dy}{dx} = 2\sqrt{x}e^{2x} - e^{2x}\frac{1}{2}x^{-\frac{1}{2}} = \frac{4xe^{2x} - e^{2x}}{2x^{\frac{3}{2}}}$

b $y = \frac{1-x^2}{e^x} \quad \frac{dy}{dx} = \frac{e^x(-2x) - (1-x^2)e^x}{e^{2x}} = \frac{x^2-2x-1}{e^x}$

c $y = \frac{e^{3x}}{1+x} \quad \frac{dy}{dx} = \frac{(1+x)3e^{3x} - e^{3x}}{(1+x)^2} = \frac{e^{3x}(2+3x)}{(1+x)^2}$

d $y = \frac{1+e^x}{1-e^x} \quad \frac{dy}{dx} = \frac{(1-e^x)e^x - (1+e^x)(-e^x)}{(1-e^x)^2} = \frac{2e^x}{(1-e^x)^2}$

4 a $y = \frac{xe^x}{1+e^x}$
 $\frac{dy}{dx} = \frac{(1+e^x)(xe^x + e^x) - xe^x(e^x)}{(1+e^x)^2} = \frac{xe^x + e^x + e^{2x}}{(1+e^x)^2}$

$$= \frac{e^x(x+1+e^x)}{(1+e^x)^2}$$

b $y = (1+e^x)^2 \quad \frac{dy}{dx} = 2e^x(1+e^x)$

c $y = \sqrt{1+e^{-x}} = (1+e^{-x})^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(1+e^{-x})^{-\frac{1}{2}}(-e^{-x}) = \frac{-e^{-x}}{2\sqrt{1+e^{-x}}}$$

d $y = \frac{x+e^x}{e^{-x}} = e^x(x+e^x)$
 $\frac{dy}{dx} = e^x(1+e^x) + e^x(x+e^x)$
 $= e^x(1+x+2e^x)$

e $y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

$$\frac{dy}{dx} = \frac{(e^x + e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2}$$

 $= \frac{-4}{(e^x - e^{-x})^2}$

5 $f(x) = x e^x \quad -3 \leq x \leq 3$

a $f'(x) = x e^x + e^x$

$$e^x(x+1) = 0 \quad \therefore x = -1 \quad y = -e^{-1}$$

\therefore one stationary point at $(-1, \frac{-1}{e})$

$$f''(x) = e^x + e^x(x+1)$$

$$f''(-1) = e^{-1} > 0 \quad \text{minimum}$$

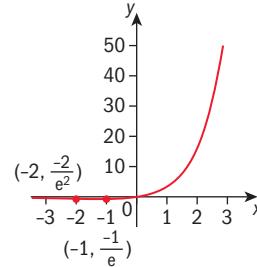
b For point of inflection $f''(x) = 0$

$$\therefore e^x(x+2) = 0 \quad \therefore x = -2 \quad y = -2e^{-2}$$

x	-3	-2	-1
$f''(x)$	$-e^{-3} < 0$	0	$e^{-1} > 0$

\therefore change of sign

\therefore point of inflection at $(-2, \frac{-2}{e^2})$



c At point of inflection $(-2, \frac{-2}{e^2})$

$$f'(-2) = -e^{-2} = -\frac{1}{e^2}$$

Equation of tangent: $y + \left(-2, \frac{-2}{e^2}\right) = -\frac{1}{e^2}(x+2)$
 $y = -\frac{1}{e^2}(x+4)$

d $y = 0, x = -4 \quad (-4, 0)$

e y -intercept = $\left(0, \frac{-4}{e^2}\right)$ area = $\frac{1}{2} \times 4 \times \frac{4}{e^2} = \frac{8}{e^2}$

Exercise 5L

1 a $y = 5^{3x} \quad \frac{dy}{dx} = (3 \ln 5) 5^{3x}$

b $y = \ln(4x+1) \quad \frac{dy}{dx} = \frac{4}{4x+1}$

2 a $y = 1 + 2 \ln x \quad \frac{dy}{dx} = \frac{2}{x}$

b $y = \frac{1}{\ln x} = (\ln x)^{-1} \quad \frac{dy}{dx} = -(\ln x)^{-2} \frac{1}{x} = \frac{-1}{x(\ln x)^2}$

Exercise 5M

1 a $y = x^2 \ln x$ $\frac{dy}{dx} = x^2 \left(\frac{1}{x}\right) + 2x \ln x = x + 2x \ln x$

b $y = xa^x$ $\frac{dy}{dx} = (x \ln a)a^x + a^x = a^x(x \ln a + 1)$

2 a $y = \ln\left(\frac{1}{x}\right) = -\ln x$ $\frac{dy}{dx} = \frac{-1}{x}$

b $y = \ln x^2 = 2 \ln x$ $\frac{dy}{dx} = \frac{2}{x}$

c $y = \frac{\ln x}{x}$ $\frac{dy}{dx} = \frac{x\left(\frac{1}{x}\right) - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$

3 a $x^y = e^x$

$$y \ln x = x$$

$$y = \frac{x}{\ln x} \quad \frac{dy}{dx} = \frac{\ln x - x\left(\frac{1}{x}\right)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$$

b $y = x^{2x}$

$$\ln y = 2x \ln x$$

$$\frac{1}{y} \frac{dy}{dx} = 2x\left(\frac{1}{x}\right) + 2 \ln x$$

$$\frac{dy}{dx} = x^{2x}(2 + 2 \ln x)$$

4 a $y = e^x(x - 1)$

$$\frac{dy}{dx} = e^x + e^x(x - 1) = xe^x$$

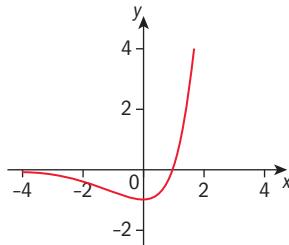
$$\frac{dy}{dx} = 0 \text{ if } x = 0 \quad \therefore \text{only one stationary point}$$

b $\frac{d^2y}{dx^2} = xe^x + e^x$

$$\text{if } x = 0, \frac{d^2y}{dx^2} = 1 > 0 \quad \therefore \text{minimum at } (0, -1)$$

c $(1, 0)$

d



5 a $x \in \mathbb{R}$

b $f(-x) = \ln(1 + (-x)^2) = \ln(1 + x^2) = f(x)$
 \therefore the y -axis is a line of symmetry

c $f(x) = \ln(1 + x^2)$

$$f'(x) = \frac{2x}{1+x^2} = 0 \quad \text{if } x = 0$$

$$f''(x) = \frac{(1+x^2)2 - 2x(2x)}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}$$

$$f''(0) = 2 > 0 \quad \therefore \text{minimum at } (0, 0)$$

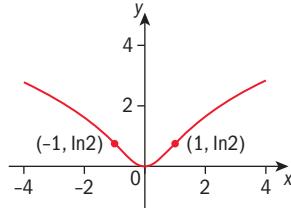
$$f''(x) = 0 \Rightarrow x = \pm 1$$

x	-2	-1	-0	1	2
$f'(x)$	$\frac{-6}{25} < 0$	0	$2 > 0$	0	$0 \frac{-6}{25} < 0$

\therefore change of sign

\therefore point of inflection at $(-1, \ln 2)$ and $(1, \ln 2)$

d



e At $(1, \ln 2)$, $f'(1) = 1$

$$\text{tangent: } y - \ln 2 = x - 1$$

$$y = x - 1 + \ln 2 \quad (\text{A})$$

$$\text{normal: } y - \ln 2 = -(x - 1)$$

$$y = -x + 1 + \ln 2 \quad (\text{B})$$

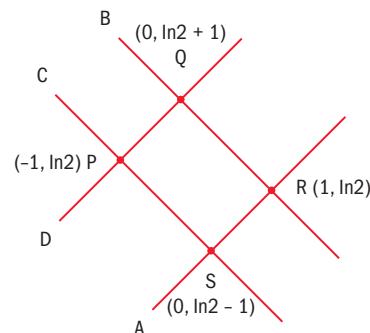
At $(-1, \ln 2)$, $f'(-1) = -1$

$$\text{tangent: } y - \ln 2 = -(x + 1)$$

$$y = -x - 1 + \ln 2 \quad (\text{C})$$

$$\text{normal: } y - \ln 2 = x + 1$$

$$y = x + 1 + \ln 2 \quad (\text{D})$$



Angles are 90° since gradients are ± 1

A and C intersection:

$$x - 1 + \ln 2 = -x - 1 + \ln 2$$

$$2x = 0$$

$$x = 0 \quad (0, \ln 2 - 1)$$

B and D intersection:

$$-x + 1 + \ln 2 = x + 1 + \ln 2$$

$$x = 0 \quad (0, \ln 2 + 1)$$

$$\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \overrightarrow{QR} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \overrightarrow{RS} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\overrightarrow{SP} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\therefore PQ = QR = RS = SP = \sqrt{2}$$

\therefore PQRS is a square, area 2

Exercise 5N

1 a arc length = $r\theta = \frac{5\pi}{4}$,

$$\text{sector area} = \frac{1}{2} r^2 \theta = \frac{25\pi}{8}$$

b arc length = $\frac{4 \times 5\pi}{12} = \frac{5\pi}{3}$,

$$\text{sector area} = \frac{1}{2} \times 16 \times \frac{5\pi}{12} = \frac{10\pi}{3}$$

- c** arc length = $5.4(2\pi - 1.3) \approx 26.9$ cm
sector area = $\frac{1}{2} \times 5.4^2(2\pi - 1.3) \approx 72.7$ cm²
- 2** Area = OPQ - OAB = $\frac{1}{2} \times 9^2 \times 0.8 - \frac{1}{2} \times 5^2 \times 0.8$
= $0.4(81 - 25) = 22.4$ m²
Perimeter = $5 \times 0.8 + 9 \times 0.8 + 4 + 4 = 19.2$ m
- 3** Length = AB + AC + BC + $3 \times \frac{1}{3}$ perimeter of circle
= $7.5 + 7.5 + 7.5 + 2\pi \times 7.5$
 ≈ 69.6 cm
- 4 a** Arc length = 5000 stadia
 $\therefore r \times \frac{7.2\pi}{180} = 5000 \therefore r = \frac{5000 \times 180}{7.2\pi}$ stadia
circumference = $2\pi r = 2\pi \times \frac{5000 \times 180}{7.2\pi}$ stadia
= $2 \times \frac{5000 \times 180}{7.2} \times 185$ m
= 46250 km
- b** Error = $\frac{46250 - 40008}{40008} \times 100\% = 13.5\%$
- 5** Let AB = x and BC = y , so AC = $\sqrt{x^2 + y^2}$
Crescent APBA + BQCB = semicircle ABA
+ semicircle BCB
- (semicircle ACA - ΔABC)
= $\frac{\pi x^2}{2} + \frac{\pi y^2}{2} - \frac{\pi}{2}(x^2 + y^2) + \Delta ABC$
= 0 + ΔABC
= ΔABC



Review exercise

- 1** $\frac{9^{2n+2} \times 6^{2n-3}}{3^{5n} \times 6 \times 4^{n-2}} = \frac{3^{4n+4} \times 6^{2n-4}}{3^{5n} \times 2^{2n-4}} = \frac{3^{4n+4} \times 3^{2n-4}}{3^{5n}}$
 $= \frac{3^{6n}}{3^{5n}} = 3^n$
- 2** $\frac{8^{\frac{2}{3}} + 4^{\frac{3}{2}}}{16^{\frac{3}{4}}} = \frac{4+8}{8} = \frac{3}{2}$
- 3 a** $9^x - 12(3^x) + 27 = 0$
 $(3^x)^2 - 12(3^x) + 27 = 0$
 $(3^x - 9)(3^x - 3) = 0$
 $3^x = 9 \quad \text{or} \quad 3^x = 3$
 $x = 1 \quad \text{or} \quad x = 2$
- b** $3^x - \frac{9}{3^x} = 8$
 $(3^x)^2 - 8(3^x) - 9 = 0$
 $(3^x - 9)(3^x + 1) = 0$
 $3^x = 9 \quad \text{or} \quad 3^x = -1$
 $x = 2$
- 4 a** $\log_a x + \log_a 3 - \log_a 7 = \log_a 12$
 $\log_a \frac{3x}{7} = \log_a 12$
 $3^x = 84$
 $x = 28$

- b** $\log_4 x - \log_4 5 = \frac{5}{2}$
 $\log_4 \frac{x}{5} = \frac{5}{2}$
 $4^{\frac{5}{2}} = \frac{x}{5}$
 $\frac{x}{5} = 32 \quad x = 160$
- c** $\log_3 x - \frac{6}{\log_3 x} = 1$
 $(\log_3 x)^2 - \log_3 x - 6 = 0$
 $(\log_3 x - 3)(\log_3 x + 2) = 0$
 $\log_3 x = 3 \quad \text{or} \quad \log_3 x = -2$
 $x = 3^3 \quad \text{or} \quad x = 3^{-2}$
 $x = 27 \quad \text{or} \quad \frac{1}{9}$
- d** $\log_7 x + 2\log 7^x = 3$
 $\log_7 x + \frac{2}{\log_7 x} = 3$
 $(\log_7 x)^2 - 3\log_7 x + 2 = 0$
 $(\log_7 x - 1)(\log_7 x - 2) = 0$
 $\log_7 x = 1 \quad \text{or} \quad 2, \quad \text{so} \quad x = 7 \quad \text{or} \quad 49$
- 5 a** $xy = 81 \quad 3\log_x y = 1 \therefore x = y^3$
 $y^4 = 81 \quad \therefore y = 3, x = 27$
- b** $y \log_2 4 = x \quad 2^x + 4^y = 512$
 $2y = x \quad 2^x + 2^{2y} = 512$
 $2^x + 2^x = 512$
 $2^x = 256$
 $\therefore x = 8 \quad y = 4$
- c** $\ln 8 + \ln(x - 6) = 2\ln y \quad 2y - x = 2$
 $\ln 8(x - 6) = \ln y^2 \quad x = 2y - 2$
 $8(x - 6) = y^2$
 $8(2y - 8) = y^2$
 $y^2 - 16y + 64 = 0$
 $(y - 8)^2 = 0 \quad \therefore y = 8 \quad x = 14$
- 6 a** $\log y + \log \frac{1}{y} = \log 1 = 0$
- b** $\frac{\log x^5 - \log x^2}{3 \log x + \log \sqrt{x}} = \frac{3 \log x}{3.5 \log x} = \frac{6}{7}$
- c** $\ln(\ln x^2) - \ln(\ln x) = \ln\left(\frac{\ln x^2}{\ln x}\right) = \ln 2$
- 7** $\log_2 x + \log_2 x^2 + \log_2 x^3 + \dots + \log_2 x^m = 3m(m+1)$
 $\log_2 x(1 + 2 + 3 + \dots + m) = 3m(m+1)$
 $\log_2 x \left(\frac{m}{2}(m+1)\right) = 3m(m+1)$
 $\log_2 x = 6$
 $x = 2^6 = 64$
- 8** $y = 5e^{2x} + 8e^{-2x} \quad \frac{dy}{dx} = 10e^{2x} - 16e^{-2x}$
 $\frac{d^2y}{dx^2} = 20e^{2x} + 32e^{-2x} = 4(5e^{2x} + 8e^{-2x})$
= 4y

9 $y = e^{3x}(2 + 5x)$

$$\frac{dy}{dx^2} = e^{3x}(5) + 3e^{3x}(2 + 5x) = e^{3x}(11 + 15x)$$

$$\frac{dy^2}{dx^2} = e^{3x}(15) + 3e^{3x}(11 + 15x) = e^{3x}(48 + 45x)$$

$$\begin{aligned}\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y &= e^{3x}(48 + 45x - 6(11 + 15x) \\ &\quad + 9(2 + 5x)) \\ &= e^{3x}(48 + 45x - 66 - 90x + 18 + 45x) \\ &= 0\end{aligned}$$

10 $e^x - e^{-x} = 4$

$$(e^x)^2 - 4e^x - 1 = 0$$

$$e^x = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$$

$$e^x > 0 \therefore e^x = 2 + \sqrt{5} \quad \therefore x = \ln(2 + \sqrt{5})$$

$$ex + e^{-x} = 2 + \sqrt{5} + \frac{1}{2 + \sqrt{5}}$$

$$= \frac{(2 \pm \sqrt{5})^2 + 1}{2 \pm \sqrt{5}} = \frac{4 + 4\sqrt{5} + 5 + 1}{2 + \sqrt{5}}$$

$$= \frac{4\sqrt{5} + 10}{2 + \sqrt{5}} = \frac{2\sqrt{5}(2 + \sqrt{5})}{2 + \sqrt{5}}$$

$$\sqrt{5}e^x + e^{-x} = 2\sqrt{5}$$

11 $f(x) = \frac{4e^x}{(e^x + 1)^2}$

$$f'(x) = \frac{(e^x + 1)^2 4e^x - 4e^x 2(e^x + 1)e^x}{(e^x + 1)^4}$$

$$= \frac{4e^x(e^x + 1)^2 - 8e^2}{(e^x + 1)^3}$$

$$= \frac{4e^x - 4e^{2x}}{(e^x + 1)^3}$$

$$f'(0) = 0 \therefore \text{stationary point at } (0, 1)$$

$$f'(x) = \frac{4e^x - 4e^{2x}}{(e^x + 1)^3}$$

$$\Rightarrow f''(x) = \frac{(e^x + 1)^3 (4e^x - 8e^{2x}) - 3(e^x + 1)^2 e^x (4e^x - 4e^{2x})}{(e^x + 1)^6}$$

$$= \frac{4e^{3x} - 16e^{2x} + 4e^x}{(e^x + 1)^4}$$

$$\therefore f''(0) = \frac{4 - 16 + 4}{16} > 0 \text{ so } (0, 1) \text{ is a maximum point}$$

Points of inflection: $f''(0) = 0$

$$\Rightarrow 4e^{3x} - 16e^{2x} + 4e^x = 0$$

$$\Rightarrow 4e^{2x} - 16e^x + 4 = 0$$

$$\Rightarrow e^{2x} - 4e^x + 1 = 0$$

$$\Rightarrow e^x = \frac{4 \pm \sqrt{12}}{2}$$

$$= 2 \pm \sqrt{3}$$

$$\Rightarrow x = \ln(2 \pm \sqrt{3})$$

12 a $f(x) = (\ln x)2 \quad x \in \mathbb{R}, x > 0$

b $f'(x) = \frac{2 \ln x}{x}$

$$f''(x) = x \left(\frac{2}{x} \right) - 2 \ln x = \frac{2 - 2 \ln x}{x^2}$$

c $f'(x) = 0 \Rightarrow \ln x = 0 \Rightarrow x = 1 \quad (1, 0)$

$$f''(1) = 2 > 0 \quad \therefore \text{minimum at } (1, 0)$$

$$f''(x) = 0 \Rightarrow \ln x = 1 \Rightarrow x = e \quad (e, 1)$$

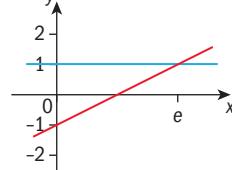
x	2	e	3
f''(x)	0.153 > 0	0	-0.0219 < 0

change of sign

\therefore point of inflection at (e, 1)

d $f'(e) = \frac{2}{e} \quad y - 1 = \frac{z}{e}(x - e)$
 $y = \frac{2}{e}x - 1$

e



$$1 = \frac{2}{e}x - 1$$

$$2 = \frac{2}{e}x$$

$$\therefore x = e$$

$$\text{area} = \frac{1}{2}(2)(e) = e$$

13 Radius of sector = AN = $a\sqrt{3}$,

$$\text{area of sector} = \frac{1}{2}$$

$$r^2 \theta = \frac{1}{2}a^2 \times 3 \times \frac{\pi}{3}$$

$$= \frac{\pi a^2}{2}$$

$$\therefore S_1 = \frac{\pi a^2}{2} - \text{area } \Delta ADE$$

$$= \frac{\pi a^2}{2} - \frac{1}{2}r^2 \sin 60^\circ$$

$$= \frac{\pi a^2}{2} - \frac{1}{2}3a^2 \times \frac{\sqrt{3}}{2}$$

$$= \frac{\pi a^2}{2} - \frac{3a^2\sqrt{3}}{4} = \frac{a^2(2\pi - 3\sqrt{3})}{4}$$

$$\frac{AE}{AC} = \frac{r}{2a} = \frac{\sqrt{3}}{2}, \text{ so } \frac{AM}{AN} = \frac{\sqrt{3}}{2}$$

$$\therefore AM = \frac{\sqrt{3}}{2} \times a\sqrt{3} = \frac{3a}{2}$$

$$\therefore S_2 = \frac{1}{2}AM^2 \theta - \Delta AFG$$

$$= \frac{1}{2} \times \frac{9a^2}{4} \times \frac{\pi}{3} - \frac{1}{2}AM^2 \sin 60^\circ$$

$$= \frac{3\pi a^2}{8} - \frac{1}{2} \times \frac{9a^2}{4} \times \frac{\sqrt{3}}{2}$$

$$= \frac{a^2(6\pi - 9\sqrt{3})}{16} = \frac{3a^2(2\pi - 3\sqrt{3})}{16}$$

$$\begin{aligned}
 \text{Similarly } S_3 &= \frac{1}{2} AC^2 \theta - \frac{1}{2} AC^2 \sin 60^\circ \\
 &= \frac{1}{2} \left(\frac{3a\sqrt{3}}{4} \right)^2 \frac{\pi}{2} - \frac{1}{2} \left(\frac{3a\sqrt{3}}{4} \right)^2 \frac{\sqrt{3}}{2} \\
 \therefore S_3 &= \frac{1}{2} \times \frac{27a^2}{16} \times \frac{\pi}{3} - \frac{1}{2} \times \frac{27a^2}{16} \times \frac{\sqrt{3}}{2} \\
 &= \frac{a^2}{64} (18\pi - 27\sqrt{3}) = \frac{9a^2}{64} (2\pi - 3\sqrt{3})
 \end{aligned}$$

Hence S_1, S_2, S_3 form a geometric series
with $r = \frac{3}{4}$.
Total area = $\frac{a}{1-r} = \frac{a^2}{4} (2\pi - 3\sqrt{3}) = a^2 (2\pi - 3\sqrt{3})$

$$\begin{aligned}
 \mathbf{14} \text{ Area required} &= \text{area of regular hexagon of side } 8 \text{ cm} - 6 \times \text{area of sector of circle} \\
 &\quad \text{of angle } 120^\circ - \text{central circle} \\
 &= 6 \times \text{area of equilateral triangle} - 6 \times \frac{\pi r^2}{3} \\
 &\quad - \pi r^2 (r = 4) \\
 &= 6 \times \frac{1}{2} \times 2r \times 2r \sin 60^\circ - 3\pi r^2 \\
 &= 6 \times \frac{1}{2} \times 64 \times \frac{\sqrt{3}}{2} - 3\pi \times 16 \\
 &= 96\sqrt{3} - 48\pi \\
 &= 15.5 \text{ cm}^2
 \end{aligned}$$