

2

Mathematics as a language

Answers

Skills check

1 $y = x^2 - 3x - 1$

$$y = \left(x - \frac{3}{2}\right)^2 - \frac{13}{4}$$

Vertex is $\left(\frac{3}{2}, -\frac{13}{4}\right)$ Axis of symmetry is $x = \frac{3}{2}$

2 a $3x + 4 = 0, x = -\frac{4}{3}$

b $3x^2 - 2x - 1 = 0$
 $(3x + 1)(x - 1) = 0$
 $x = -\frac{1}{3}$ or 1

3 a $y - 3 = \sqrt{x - 2}$
 $\therefore x - 2 = (y - 3)^2$
 $\therefore x = (y - 3)^2 + 2$

b $y = \frac{2x - 1}{3x + 2}$
 $3xy + 2y = 2x - 1$
 $2x - 3xy = 2y + 1$
 $x(2 - 3y) = 2y + 1$
 $x = \frac{2y + 1}{2 - 3y}$

Exercise 2A

1 a function, domain = {0, 1, 2, 3},
range = {-1, 1, 2, 3}

b function, domain = {-3, -2, -1, 0},
range = {0}

c not a function

d not a function

2 a function, domain = {x | -3 ≤ x ≤ 3},
range = {y | 0 ≤ y ≤ 3}

b not a function

c not a function

d function, domain = {x | x ≥ -1},
range = {y | y ≥ 0}

Exercise 2B

1 $y^2 = x \Rightarrow y = \pm \sqrt{x}$

one value of x gives 2 values of y
eg. if $x = 4, y = \pm 2 \therefore$ not a function.

$y = \sqrt{x}, \sqrt{x}$ is the positive square root of x

\therefore each value of x gives just one value of y

- 2 a $y = x^2 - 4x + 2$ domain is $x \in \mathbb{R}$
 $y = (x - 2)^2 - 2$ range = $\{y | y \geq -2\}$
- b $y = -(x + 2)^2 - 3$ domain is $x \in \mathbb{R}$
range = $\{y | y \leq -3\}$
- c $y = \sqrt{x + 2}$ $x + 2 \geq 0$ domain = $\{x | x \geq -2\}$
 $x \geq -2$ range = $\{y | y \geq 0\}$
- d $y = \sqrt{3 - x}$ $3 - x \geq 0$ domain = $\{x | x \leq 3\}$
 $x \leq 3$ range = $\{y | y \geq 0\}$
- e $y = -3x^2 + 6x - 1$ domain is $x \in \mathbb{R}$
 $= -3(x^2 - 2x) - 1$
 $= -3[(x - 1)^2 - 1] - 1$
 $= -3(x - 1)^2 + 2$ range = $\{y | y \leq 2\}$
- f $y = \sqrt{4 - 2x}$ $4 - 2x \geq 0$ domain = $\{x | x \leq 2\}$
 $x \leq 2$ range = $\{y | y \geq 0\}$

Exercise 2C

1 $y = -|x|$ domain = $\{x | x \in \mathbb{R}\}$
range = $\{y | y \leq 0\}$

2 $y = |2x + 1|$ domain = $\{x | x \in \mathbb{R}\}$
range = $\{y | y \geq 0\}$

3 $y = -|2x + 1|$ domain = $\{x | x \in \mathbb{R}\}$
range = $\{y | y \leq 0\}$

4 $y = 2|x - 1|$ domain = $\{x | x \in \mathbb{R}\}$
range = $\{y | y \geq 0\}$

5 $y = -\frac{1}{2}|3x + 2|$ domain = $\{x | x \in \mathbb{R}\}$
range = $\{y | y \leq 0\}$

6 $y = |x + 4| - 2$ domain = $\{x | x \in \mathbb{R}\}$
range = $\{y | y \geq -2\}$

7 $y = -2|x - 1| + 1$ domain = $\{x | x \in \mathbb{R}\}$
range = $\{y | y \leq 1\}$

8 $y = 3|1 - 2x| - 2$ domain = $\{x | x \in \mathbb{R}\}$
range = $\{y | y \geq -2\}$

Exercise 2D

1 $y = \frac{1}{3x + 2}$ domain = $\{x | x \neq -\frac{2}{3}\}$,
range = $\{y | y \neq 0\}$

2 $y = -\frac{1}{2-x}$ domain = $\{x | x \neq 2\}$,
range = $\{y | y \neq 0\}$

3 $y = \frac{3}{3-x}$ domain = $\{x | x \neq 3\}$,
range = $\{y | y \neq 0\}$

4 $y = -\frac{5}{6x+3}$ domain = $\left\{x \mid x \neq -\frac{1}{2}\right\}$, range = $\{y \mid y \neq 0\}$

5 $y = \frac{1+2x}{1-2x}$ domain = $\left\{x \mid x \neq \frac{1}{2}\right\}$, range = $\{y \mid y \neq -1\}$

6 $y = -\frac{2-3x}{1+x}$ domain = $\{x \mid x \neq -1\}$, range = $\{y \mid y \neq 3\}$

Exercise 2E

1 $y = \frac{1}{|x+1|}$ domain = $\{x \mid x \neq -1\}$, range = $\{y \mid y > 0\}$

2 $y = \frac{-2}{|x-1|}$ domain = $\{x \mid x \neq 1\}$, range = $\{y \mid y < 0\}$

3 $y = \frac{x}{|x|}$ domain = $\{x \mid x \neq 0\}$

If $x > 0$, $y = \frac{x}{x} = 1$

If $x < 0$, $y = \frac{x}{-x} = -1$

\therefore range = $\{-1, 1\}$

4 $y = \frac{-2}{\sqrt{1-x}}$ $1-x > 0 \quad \therefore x < 1$
domain = $\{x \mid x < 1\}$, range = $\{y \mid y < 0\}$

5 a For f to be real, $\sqrt{\frac{1}{x^2}-2} > 0$

$$\Rightarrow \frac{1}{x^2} - 2 > 0$$

$$\Rightarrow \frac{1}{x^2} > 2$$

$$\therefore x^2 < \frac{1}{2}$$

$$\text{so domain} = \left\{x \mid -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}, x \neq 0\right\}$$

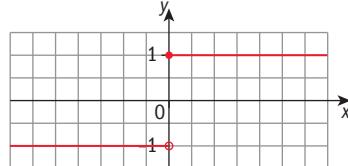
b Range = $\{y \mid y > 0\}$

Exercise 2F

1 $y = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$

a $f(-3) = -1 \quad f(0) = 1 \quad f(\pi) = 1 \quad f(4) = 1$

b

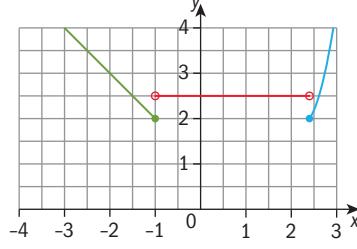


c domain = $\{x \mid x \in \mathbb{R}\}$ range = $\{-1, 1\}$

2 $y = \begin{cases} 1-x, & x \leq -1 \\ 2.5, & -1 < x < \sqrt{6} \\ x^2 - 4, & x \geq \sqrt{6} \end{cases}$

a $f(-3) = 4 \quad f(0) = 2.5 \quad f(\sqrt{6}) = 2 \quad f(3) = 5$

b

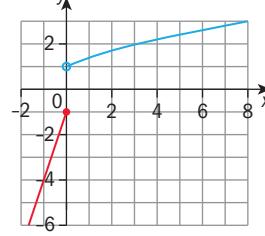


c domain = $\{x \mid x \in \mathbb{R}\}$ range = $\{y \mid y \geq 2\}$

3 $f(x) = \begin{cases} 3x-1, & x \leq 0 \\ \sqrt{x+1}, & x > 0 \end{cases}$

a $f(-1) = -4 \quad f(0) = -1 \quad f(1) = \sqrt{2} \quad f(8) = 3$

b

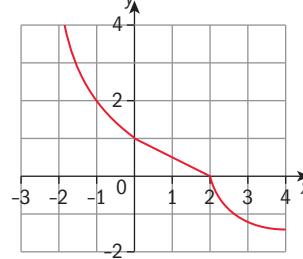


c domain = $\{x \mid x \in \mathbb{R}\}$ range = $\{y \mid y \leq -1 \text{ or } y > 1\}$

4 $g(x) = \begin{cases} x^2 + 1, & x \leq 0 \\ -\frac{1}{2}x + 1, & 0 < x \leq 2 \\ -\sqrt{x-2}, & x > 2 \end{cases}$

a $f(-2) = 5 \quad f(1) = \frac{1}{2} \quad f(2) = 0 \quad f(3) = -1$

b



c domain = $\{x \mid x \in \mathbb{R}\}$ range = $\{y \mid y \in \mathbb{R}\}$

Exercise 2G

1 $f(x) = 4 - x^2$ a many-to-one

b $f(-x) = 4 - (-x)^2 = 4 - x^2 = f(x)$ \therefore even

2 $g(x) = x^3 + 3x$ a one-to-one

b $g(-x) = (-x)^3 + 3(-x) = -x^3 - 3x = -g(x)$
 \therefore odd

3 $h(x) = \frac{-3}{2x}$ a one-to-one

b $h(-x) = \frac{-3}{2(-x)} = \frac{3}{2x} = -h(x)$ \therefore odd

4 $p(x) = x^3 + 4x + 1$ a one-to-one

b $p(-x) = (-x)^3 + 4(-x) + 1 = -x^3 - 4x + 1 \neq p(x) \text{ or } -p(x)$ \therefore neither odd nor even

5 $r(x) = \begin{cases} -1 & 0 \leq x < \pi \\ 1 & \pi \leq x < 2\pi \\ -1 & 2\pi < x < 3\pi \end{cases}$

a many-to-one
b If $0 \leq x < 3\pi$, $r(-x)$ is not defined
 \therefore neither even nor odd

6 $q(x) = 2x^3 - 4x$ a many-to-one

b $q(-x) = 2(-x)^3 - 4(-x) = -2x^3 + 4x = -q(x)$
 \therefore odd

- 7 $w(x) = x - 2x^3 + x^5$ a many-to-one
b $w(-x) = -x - 2(-x)^3 + (-x)^5 = -x + 2x^3 - x^5 = -w(x)$
 \therefore odd
- 8 $t(x) = 4x^4 - x$ a many-to-one
b $t(-x) = 4(-x)^4 - (-x) = 4x^4 + x \neq t(x)$ or $-t(x)$
 \therefore neither even nor odd
- 9 $f(x) = 0$ is both even and odd

Exercise 2H

- 1 $f(x) = 2x$ $g(x) = \sqrt{x}$
domain of f is all real numbers
domain of g is all non-negative real numbers
a $2g(x) - f(x)$ domain is all non-negative real numbers
b $f(x) \cdot g(x)$ domain is all non-negative real numbers
c $\left(\frac{g}{f}\right)(x)$ domain is all positive real numbers

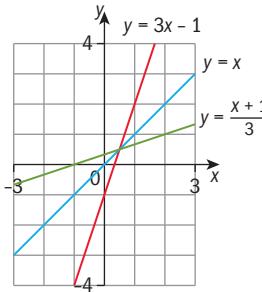
- 2 $f(x) = |x + 1|$ $g(x) = \sqrt{x^2 - 4}$
domain of f is all real numbers
domain of $g = \{x | x \leq -2, x \geq 2\}$
domain of $\left(\frac{f}{g}\right)(x) = \{x | x < -2, x > 2\}$
- 3 $f(x) = x^2 + 2x - 1$ $g(x) = 1 - 2x - 3x^2$
a $f(g(0)) = f(1) = 2$
b $g(f(-1)) = g(-2) = -7$
c $f(f(0)) = f(-1) = -2$
d $g(g(x)) = g(1 - 2x - 3x^2)$
 $= 1 - 2(1 - 2x - 3x^2) - 3(1 - 2x - 3x^2)^2$
 $= 1 - 2 + 4x + 6x^2 - 3(1 - 2x - 3x^2 - 2x + 4x^2 + 6x^3 - 3x^2 + 6x^3 + 9x^4)$
 $= -1 + 4x + 6x^2 - 3 + 12x + 6x^2 - 36x^3 - 27x^4$
 $= -4 + 16x + 12x^2 - 36x^2 - 27x^4$

- 4 $f(x) = 1 - 2x$ $g(x) = x^2 - 1$ $h(x) = \sqrt{2x + 4}$
a, b i $f(g(x)) = f(x^2 - 1)$
 $= 1 - 2(x^2 - 1) = 3 - 2x^2$
domain = $\{x | x \in \mathbb{R}\}$ range = $\{y | y \leq 3\}$
ii $g(h(x)) = g(\sqrt{2x + 4}) = 2x + 4 - 1 = 2x + 3$ domain
= domain of $h = \{x | x \geq -2\}$ range
= $\{y | y \geq -1\}$
iii $f(h(x)) = f(\sqrt{2x + 4}) = 1 - 2\sqrt{2x + 4}$
domain = $\{x | x \geq -2\}$ range = $\{y | y \leq 1\}$
iv $h(g(x)) = h(x^2 - 1) = \sqrt{2(x^2 - 1) + 4} = \sqrt{2x^2 + 2}$
domain = $\{x | x \in \mathbb{R}\}$ range = $\{y | y \geq \sqrt{2}\}$
d $f(g(1)) = 3 - 2(1)^2 = 1$
 $h(f(g(1))) = h(1) = \sqrt{6}$

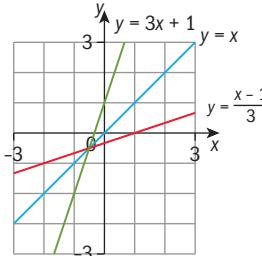
- 5 e.g. $f(x) = x - 2$ $g(x) = x^2$
6 e.g. $g(x) = 2x - 3$ $h(x) = \sqrt{x}$

Exercise 2I

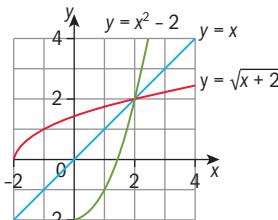
- 1 $y = 3x - 1$ $x = 3y - 1$
 $\therefore y = \frac{x+1}{3}$ $f^{-1}(x) = \frac{x+1}{3}$
 $f(f^{-1}(x)) = 3\left(\frac{x+1}{3}\right) - 1 = x + 1 - 1 = x$
 $f^{-1}(f(x)) = \frac{(3x-1)+1}{3} = \frac{3x}{3} = x$



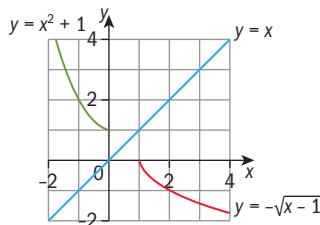
- 2 $y = \frac{x-1}{3}$
 $x = \frac{y-1}{3}$ $\therefore y = 3x + 1$ $f^{-1}(x) = 3x + 1$
 $f(f^{-1}(x)) = \frac{(3x+1)-1}{3} = \frac{3x}{3} = x$



- 3 $y = x^2 - 2, x \geq 0$
 $x = y^2 - 2$ $\therefore y = \sqrt{x+2}, x \geq -2$
 $f^{-1}(x) = \sqrt{x+2}, x \geq -2$
 $f(f^{-1}(x)) = (\sqrt{x+2})^2 - 2 = x + 2 - 2 = x$
 $f^{-1}(f(x)) = \sqrt{x^2 - 2 + 2} = \sqrt{x^2} = x$



- 4 $y = x^2 + 1$ $x \leq 0$
 $x = y^2 + 1$
 $\therefore y = -\sqrt{x-1}, x \geq 1$ $f^{-1}(x) = -\sqrt{x-1}, x \geq 1$
 $f(f^{-1}(x)) = (-\sqrt{x+1})^2 + 1 = x - 1 + 1 = x$
 $f^{-1}(f(x)) = -\sqrt{x^2 + 1 - 1} = -\sqrt{x^2} = x$



5 $y = x^2 + 4x - 1 \quad x \geq -1$

$$x = y^2 + 4y - 1$$

$$y^2 + 4y = x + 1$$

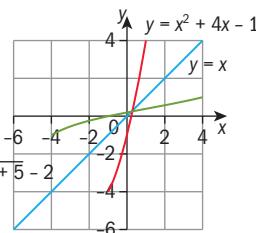
$$y^2 + 4y + 4 = x + 5$$

$$(y+2)^2 = x + 5$$

$$y + 2 = \sqrt{x+5}$$

$$y = \sqrt{x+5} - 2$$

$$f^{-1}(x) = \sqrt{x+5} - 2, \quad x \geq -4 \quad (\text{since range of } f(x) \text{ is } y \geq -4)$$



6 $y = 1 - 2x \quad x = 1 - 2y$

$$2y = 1 - x$$

$$y = \frac{1-x}{2}$$

$\therefore y = 1 - 2x$ is not its own inverse

7 $f(x) = 3x \quad g(x) = 2x + 1$

$$f^{-1}(x) = \frac{x}{3} \quad g^{-1}(x) = \frac{x-1}{2}$$

$$g \circ f(x) = g(3x) = 2(3x) + 1 = 6x + 1$$

$$(g \circ f)^{-1}(x) = \frac{x-1}{6} \quad f^{-1} \circ g^{-1}(x) = f^{-1}\left(\frac{x-1}{2}\right) = \frac{x-1}{6}$$

$$\therefore f^{-1} \circ g^{-1}(x) = (g \circ f)^{-1}(x)$$

8 $y = \frac{2x+1}{x-1}$

$$x = \frac{2y+1}{y-1}$$

$$y \cdot x - x = 2y + 1$$

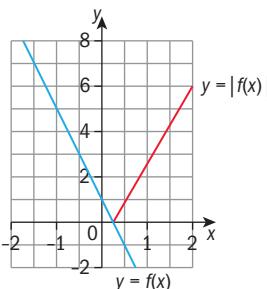
$$y(x-2) = 1+x$$

$$y = \frac{1+x}{x-2}$$

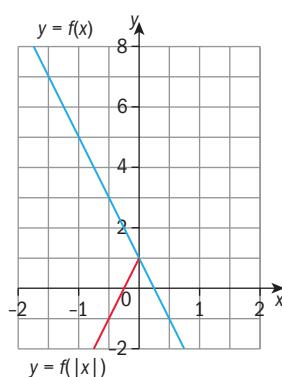
$$\therefore f^{-1}(x) = \frac{1+x}{x-2}, \text{ domain} = \{x \mid x \neq 2\}$$

Exercise 2J

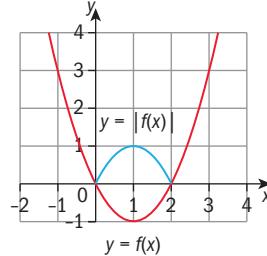
1 a



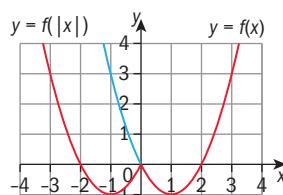
b



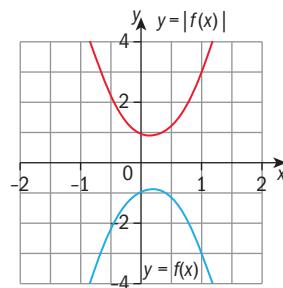
2 a



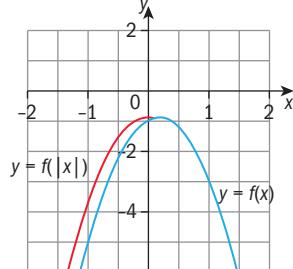
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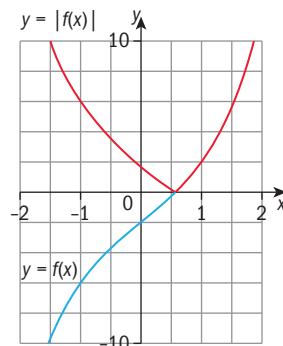
3 a



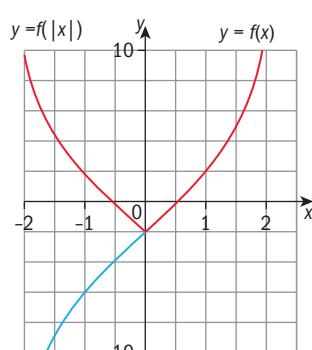
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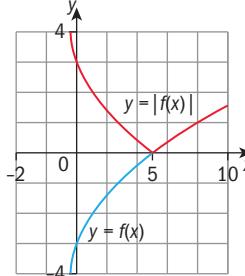
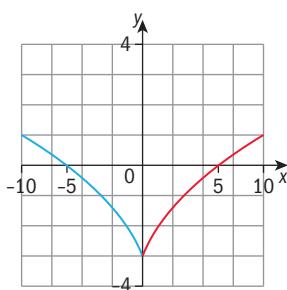
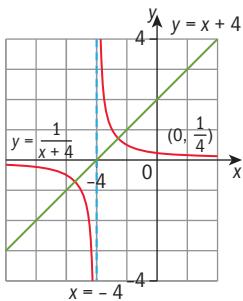
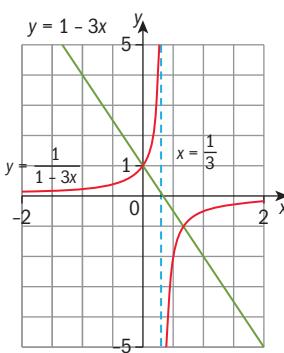
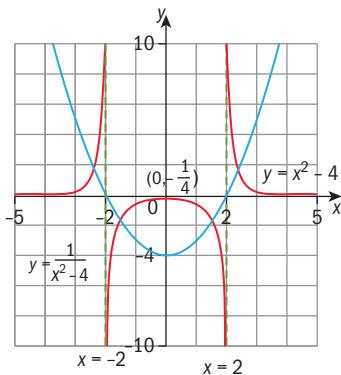
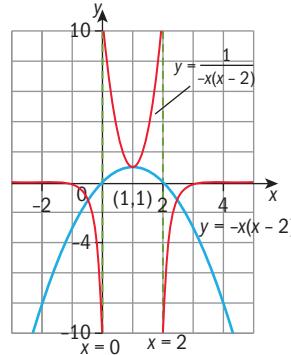
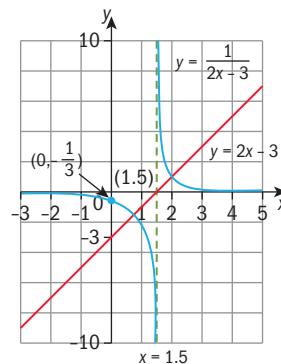
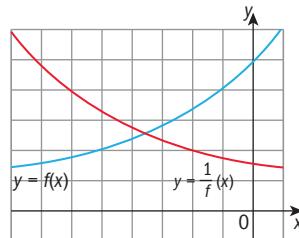
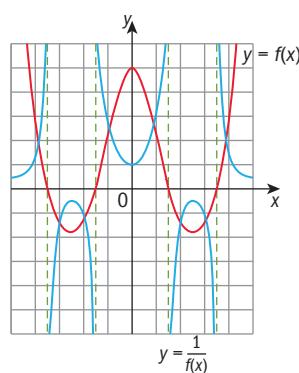


4 a



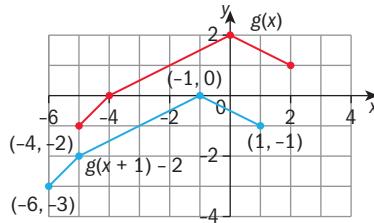
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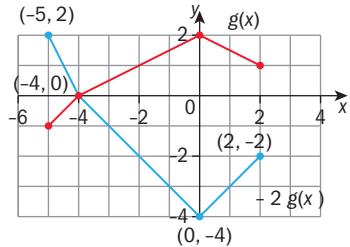
5 a**b****Exercise 2K****1****2****3****4****5****6 a****b****Exercise 2L****1 a** Translation $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, i.e. $g(x) = f(x - 3) + 2$ **b** Translation $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$, i.e. $g(x) = f(x + 2) - 1$ **c** Reflection in the x -axis and translation $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, i.e. $g(x) = -f(x) - 1$ **d** Horizontal compression by a factor of $\frac{1}{2}$, i.e. $g(x) = f(2x)$ **e** Reflection in the x -axis and vertical stretch of 2, i.e. $g(x) = -2f(x)$ **f** Reflection in the y -axis and horizontal stretch of 2, i.e. $g(x) = f\left(-\frac{1}{2}x\right)$

2 $g(x) = h(-(x - 3))$ or $g(x) = h(-x + 3)$

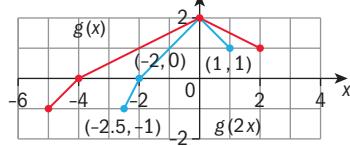
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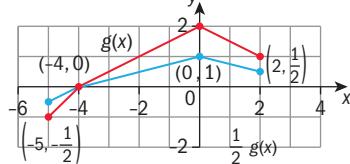
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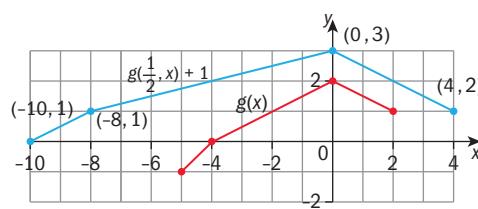
c



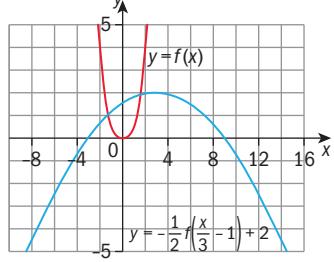
d



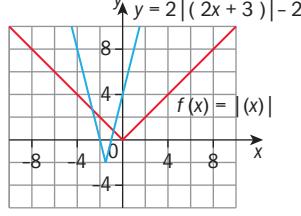
e



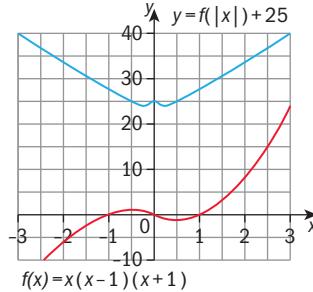
4 a



b



c



5 $y = \frac{2}{3(x+2)} + 3$ $y = \frac{2+9(x+2)}{3(x+2)}$ $y = \frac{9x+20}{3(x+2)}$

Domain = $\{x \mid x \neq -2\}$, Range = $\{y \mid y \neq 3\}$

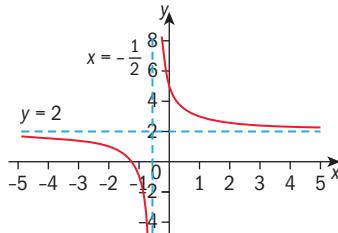
6 a $x = -\frac{1}{2}, y = 2$

b When $x = 0, y = \frac{5}{1} = 5$
 \Rightarrow intercept at $(0, 5)$

When $y = 0, 4x + 5 = 0$

$$\begin{aligned} 4x &= -5 \\ x &= -\frac{5}{4} \\ \Rightarrow \text{intercept at } &\left(-\frac{5}{4}, 0\right) \end{aligned}$$

c



d $y = \frac{2(2x+1)+3}{2x+1} = 2 + \frac{3}{2x+1}$

\therefore if $g(x) = \frac{1}{x}$, then $f(x) = 3g(2x+1) + 2$
i.e. vertical stretch of 3, then horizontal
stretch of $\frac{1}{2}$, then translation $\begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix}$.



Review exercise

1 a function domain = $\{x \mid x \in \mathbb{R}\}$
range = $\{y \mid y \leq 2\}$

b not a function

c not a function

d function domain = $\{x \mid x \in \mathbb{R}, x \neq \pm 1\}$
range = $\{y \mid y \leq -0.25 \text{ or } y > 0\}$

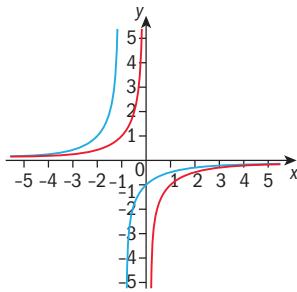
2 $f(g(h(3))) = f(g(4)) = f(2) = 3$
 $h^{-1}(g^{-1}(f^{-1}(3))) = h^{-1}(g^{-1}(2)) = h^{-1}(4) = 3$

3 $f \circ f(x) = f\left(\frac{1}{x-1}\right) = \frac{1}{\frac{1}{x-1}-1} = \frac{x-1}{1-(x-1)} = \frac{x-1}{2-x}$
 $y = \frac{x-1}{2-x} \quad x = \frac{y-1}{2-y} \quad 2x - yx = y - 1$

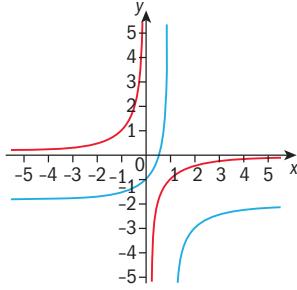
$$\begin{aligned} y + yx &= 2x + 1 \quad y(1+x) = 2x + 1 \quad y = \frac{2x+1}{x+1} \\ \therefore (f \circ f)^{-1}(x) &= \frac{2x+1}{x+1} \end{aligned}$$

- 4 $(-5, -2) \rightarrow (-6, 1)$ $(-4, 0) \rightarrow (-5, -3)$
 $(-3, 2) \rightarrow (-4, -7)$ $(-1, -1) \rightarrow (-2, -1)$
 $(3, -3) \rightarrow (2, 3)$ $(8, 2) \rightarrow (7, -7)$

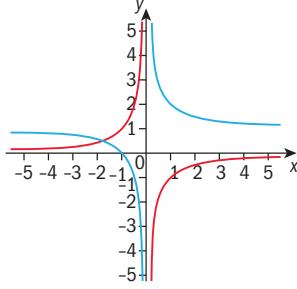
5 a



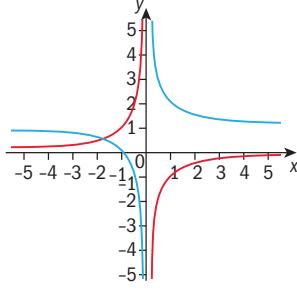
b



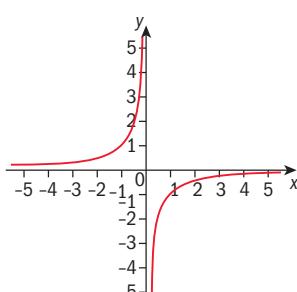
c



d



e



- 6 a Reflection in x -axis $\Rightarrow g(x) = -f(x)$
 b Reflection in y -axis $\Rightarrow g(x) = f(-x)$
 c Horizontal shift of 3 units to the left, vertical shift of 1 unit down $\Rightarrow g(x) = f(x + 3) - 1$
 d Reflection in the horizontal line $y = 1$
 $\Rightarrow g(x) = -f(x) + 1$
 e $g(x) = \frac{1}{f(x)}$

f Compression by factor of $\frac{1}{2}$ and reflection in x -axis $\Rightarrow g(x) = -f(2x)$

g Reflection in y -axis and vertical shift of 2 units upwards $\Rightarrow g(x) = f(-x) + 2$

7 $f(x)$ is odd, so $f(-x) = -f(x)$

Let $x = 0$, then $f(0) = -f(0)$

i.e. $2f(0) = 0$

$\therefore f(0) = 0$

$\therefore f(x)$ passes through $(0, 0)$



Review exercise

1 a $f(g(x)) = f(x^2) = 3x^2 - 1$

b $h(g(x)) = h(x^2) = \frac{1}{x^2 + 2}$

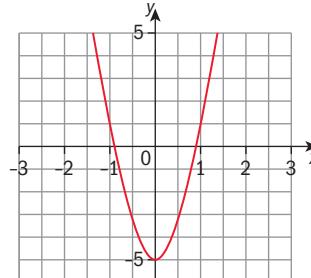
c $g^{-1}(x) = \sqrt{x}$ $g^{-1}(f(x))$

$$= g^{-1}(3x - 1) = \sqrt{3x - 1}$$

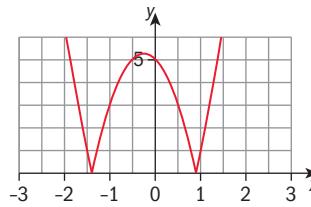
d Find h^{-1} : $x = \frac{1}{y+2} \therefore y + 2 = \frac{1}{x} \therefore y = \frac{1}{x} - 2$
 $f(h^{-1}(x)) = f\left(\frac{1}{x} - 2\right) = 3\left(\frac{1}{x} - 2\right) - 1 = \frac{3}{x} - 7$
 $= \frac{3 - 7x}{x}$

2 $f(x) = \frac{x-1}{x+3}$ $g(x) = x^2$

3 $f(x) = |x|$ $g(x) = 4x^2 + 2x - 5$
 $g(f(x)) = 4|x|^2 + 2|x| - 5$



$f(g(x)) = |4x^2 + 2x - 5|$



4 Vertical stretch of $\frac{9}{5}$ followed by translation $\begin{pmatrix} 0 \\ 32 \end{pmatrix}$
 $x = \frac{5}{9}x - \frac{160}{9}$

$$9x = 5x - 160$$

$$4x = -160 \quad \text{and} \quad \therefore x = -40$$

5 $g(x) = \frac{3}{2(x+1)} - 2$
 $= \frac{3 - 4(x+1)}{2(x+1)} = \frac{-1 - 4x}{2(x+1)}$