

## 1

# Mathematics as the science of patterns

## Answers

### Skills check

**1** **a**  $\{1, 2, 3, 4, 5\}$  **b**  $\{-4, -3, -2, -1, 0, 1\}$

**c**  $\{1, 2, 3, 4, 5, 6\}$

**2** **a**  $3(x - 4) - 2(x + 7) = 0$

$$3x - 12 - 2x - 14 = 0$$

$$x = 26$$

**b**  $3x - 2(2x + 5) = 2$

$$3x - 4x - 10 = 2$$

$$-x = 12$$

$$x = -12$$

**c**  $5x + 4 - 2(x + 6) = x - (3x - 2)$

$$5x + 4 - 2x - 12 = x - 3x + 2$$

$$3x - 8 = -2x + 2$$

$$5x = 10$$

$$x = 2$$

**3** **a**  $2(\sqrt{3} - 2) + \sqrt{3}(1 - \sqrt{3}) = 2\sqrt{3} - 4 + \sqrt{3} - 3$   
 $= 3\sqrt{3} - 7$

**b**  $\frac{3}{\sqrt{2}} + 5\sqrt{2} = \frac{3\sqrt{2}}{2} + 5\sqrt{2} = \frac{13}{2}\sqrt{2}$

**c**  $\frac{(1+\sqrt{3})}{(1-\sqrt{3})} = \frac{(1+\sqrt{3})(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})} = \frac{1+\sqrt{3}+\sqrt{3}+3}{1-3}$   
 $= \frac{4+2\sqrt{3}}{-2} = -2-\sqrt{3}$

**4** **a**  $\frac{1}{(x-2)} = \frac{-3}{(1-2x)}$   
 $1 - 2x = -3(x - 2)$   
 $1 - 2x = -3x + 6$   
 $x = 5$

**b**  $\frac{2x}{2x^2+1} = \frac{1}{x-1}$   
 $2x(x-1) = 2x^2 + 1$   
 $2x^2 - 2x = 2x^2 + 1$   
 $-2x = 1$   
 $x = -\frac{1}{2}$

**5** **a** 35                   **b** -10

### Exercise 1A

**1** **a** 0, 1.5, 3

**b**  $\frac{9}{10}, \frac{11}{12}, \frac{13}{14}$

**c**  $\frac{1}{99}, \frac{1}{143}, \frac{1}{195}$

(denominators can be written as  $1 \times 3, 3 \times 5, 5 \times 7, 7 \times 9, 9 \times 11, 11 \times 13, 13 \times 15$ )

**2** **a**  $r(r + 1)$

**b**  $\frac{1}{r^2+1}$

**c**  $2r - 3$

**3** **a** 1, 5, 9, 13

**b**  $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}$

**c**  $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$

**4** **a**  $2 + 6 + 12 + 20$

**b**  $\frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \frac{4}{9} + \frac{5}{11}$

**c**  $-1 + 4 - 9 + 16 - 25$

**5** **a**  $\sum_{r=1}^{\infty} 4r - 5$

**b**  $\sum_{r=1}^{10} (-1)^r$

**c**  $\sum_{r=1}^6 6(-2)^{r-1}$

### Investigation - quadratic sequences

**N = n<sup>2</sup> - 2n + 3**

$$\begin{aligned} n = p - 1 \Rightarrow n^2 - 2n + 3 &= (p - 1)^2 - 2(p - 1) + 3 \\ &= p^2 - 2p + 1 - 2p + 2 + 3 \\ &= p^2 - 4p + 6 \end{aligned}$$

$$n = p \Rightarrow n^2 - 2n + 3 = p^2 - 2p + 3$$

$$\begin{aligned} n = p + 1 \Rightarrow n^2 - 2n + 3 &= (p + 1)^2 - 2(p + 1) + 3 \\ &= p^2 + 2p + 1 - 2p - 2 + 3 \\ &= p^2 + 2 \end{aligned}$$

first differences are  $2p - 3$  and  $2p - 1$

second difference =  $(2p - 1) - (2p - 3) = 2$  (a constant)

**N = 2n<sup>2</sup> + 2n + 1**

$$\begin{aligned} n = p - 1 \Rightarrow 2n^2 + 2n + 1 &= 2(p - 1)^2 + 2(p - 1) + 1 \\ &= 2p^2 - 4p + 2 + 2p - 2 + 1 \\ &= 2p^2 - 2p + 1 \end{aligned}$$

$$n = p \Rightarrow 2n^2 + 2n + 1 = 2p^2 + 2p + 1$$

$$\begin{aligned} n = p + 1 \Rightarrow 2n^2 + 2n + 1 &= 2(p + 1)^2 + 2(p + 1) + 1 \\ &= 2p^2 + 4p + 2 + 2p + 2 + 1 \\ &= 2p^2 + 6p + 5 \end{aligned}$$

first differences are  $4p$  and  $4p + 4$

second difference = 4 (a constant)

**N = -n<sup>2</sup> + 3n - 4**

$$\begin{aligned} n = p - 1 \Rightarrow -n^2 + 3n - 4 &= -(p-1)^2 + 3(p-1) - 4 \\ &= -p^2 + 2p - 1 + 3p - 3 - 4 \\ &= -p^2 + 5p - 8 \end{aligned}$$

$$n = p \Rightarrow -n^2 + 3n - 4 = -p^2 + 3p - 4$$

$$\begin{aligned} n = p + 1 \Rightarrow -n^2 + 3n - 4 &= -(p+1)^2 + 3(p+1) - 4 \\ &= -p^2 - 2p - 1 + 3p + 3 - 4 \\ &= -p^2 + p - 2 \end{aligned}$$

first differences are  $-2p + 4$  and  $-2p + 2$

$$\begin{aligned} \text{second difference} &= (-2p + 2) - (-2p + 4) \\ &= -2 \text{ (a constant)} \end{aligned}$$

Conjecture: For the quadratic  $N = an^2 + bn + c$  the second difference is a constant and is equal to  $2a$ .

Proof:

$$\begin{aligned} n = p - 1 \Rightarrow an^2 + bn + c &= a(p-1)^2 + b(p-1) + c \\ &= ap^2 - 2ap + a + bp - b + c \end{aligned}$$

$$n = p \Rightarrow an^2 + bn + c = ap^2 + bp + c$$

$$\begin{aligned} n = p + 1 \Rightarrow an^2 + bn + c &= a(p+1)^2 + b(p+1) + c \\ &= ap^2 + 2ap + a + bp + b + c \end{aligned}$$

first differences are  $2ap - a + b$  and  $2ap + a + b$

second difference =  $2a$ , which proves the conjecture.

**Investigation – triangular numbers**

Since the second difference is a constant (1) the triangle numbers can be generated by a quadratic

$$N = an^2 + bn + c \quad 2a = 1 \Rightarrow a = \frac{1}{2}$$

$$N = \frac{1}{2}n^2 + bn + c$$

$$n = 1 \Rightarrow \frac{1}{2} + b + c = 1 \Rightarrow b + c = \frac{1}{2}$$

$$n = 2 \Rightarrow 2 + 2b + c = 3 \Rightarrow 2b + c = 1$$

$$\therefore b = \frac{1}{2}, c = 0$$

$$N = \frac{1}{2}n^2 + \frac{1}{2}n \quad \text{or} \quad N = \frac{1}{2}n(n+1)$$

**Investigation – more number patterns**

Square numbers:  $N = n^2$

Pentagonal numbers:  $N = \frac{n(3n-1)}{2}$

Hexagonal numbers:  $N = n(2n-1)$

Heptagonal numbers:  $N = \frac{n(5n-3)}{2}$

Polygonal numbers	<b>N</b>
triangle	$\frac{1}{2}n(n+1) = \frac{n}{2}(n+1)$
square	$n^2 = \frac{n}{2}(2n+0)$
pentagon	$\frac{n(3n-1)}{2} = \frac{n}{2}(3n-1)$
hexagon	$n(2n-1) = \frac{n}{2}(4n-2)$
heptagon	$\frac{n(5n-3)}{2} = \frac{n}{2}(5n-3)$

Conjecture: For a polygon with  $k$  sides the polygonal numbers are given by

$$N = \frac{n}{2} [(k-2)n - (k-4)]$$

**Exercise 1B**

$$\mathbf{1} \quad \mathbf{a} \quad u_n = 5 + (n-1)6$$

$$u_n = 6n - 1$$

$$\mathbf{b} \quad u_n = 10 + (n-1)(-7)$$

$$u_n = -7n + 17$$

$$\mathbf{c} \quad u_n = a + (n-1)2$$

$$u_n = 2n + a - 2$$

$$\mathbf{2} \quad \mathbf{a} \quad u_{15} = 2 + 14d = 2 + 14 \times 9 = 128$$

$$\mathbf{b} \quad u_{12} = -1 + 11d = -1 + 11 \times \frac{5}{4} = \frac{51}{4}$$

$$\mathbf{c} \quad u_n = 3 + (n-1)4 = 4n - 1$$

$$\mathbf{3} \quad a + 3d = 18 \Rightarrow a - 15 = 18 \Rightarrow a = 33$$

$$u_n = 33 + (n-1)(-5) = 38 - 5n$$

$$\mathbf{4} \quad a + 3d = 0 \quad (1)$$

$$a + 13d = 40 \quad (2)$$

$$(2) - (1) \Rightarrow 10d = 40 \Rightarrow d = 4$$

$$\therefore a + 12 = 0 \text{ and } a = -12$$

$$\mathbf{5} \quad \text{Salary after 15 years} = u_{16} = a + 15d$$

$$= 48000 + 15 \times 500$$

$$= € 55500$$

$$\text{Need } n \times 500 = 24000$$

$$\Rightarrow n = 48 \text{ years}$$

**Exercise 1C**

$$\mathbf{1} \quad \mathbf{a} \quad u_1 = 6 \quad d = 13 \quad u_n = 110$$

$$6 + (n-1)13 = 110$$

$$(n-1)13 = 104$$

$$n-1 = 8$$

$$n = 9$$

$$S_9 = \frac{9}{2}(6 + 110) = 522$$

$$\mathbf{b} \quad u_1 = 52 \quad d = -11 \quad u_n = -25$$

$$52 + (n-1)(-11) = -25$$

$$(n-1)(-11) = -77$$

$$n-1 = 7$$

$$n = 8$$

$$S_8 = \frac{8}{2}(52 - 25) = 108$$

$$\mathbf{c} \quad u_1 = -78 \quad d = -4 \quad u_n = -142$$

$$-78 + (n-1)(-4) = -142$$

$$(n-1)(-4) = -64$$

$$n-1 = 16$$

$$n = 17$$

$$S_{17} = \frac{17}{2}(-78 - 142) = -1870$$

$$\mathbf{2} \quad \mathbf{a} \quad \sum_{r=1}^{10} 5r + 7 = 12 + 17 + 22 + \dots + 57$$

$$= \frac{10}{2}(12 + 57)$$

$$= 345$$

**b**  $\sum_{r=1}^{15} 5 - 3r = 2 - 1 - 4 \dots - 40$   
 $= \frac{15}{2} (2 - 40)$   
 $= -285$

**3**  $u_1 = 60 \quad u_{10} = -3 \quad n = 16$   
 $60 + 9d = -3$   
 $9d = -63$   
 $d = -7$   
 $S_{16} = \frac{16}{2} (2 \times 60 + 15 \times -7) = 120$

**4**  $S_5 = 25 \quad u_4 = 8$   
 Let the numbers be  
 $u - 2d, u - d, u, u + d, u + 2d$   
 $S_5 = u - 2d + u - d + u + u + d + u + 2d$   
 $\therefore 5u = 25$   
 $u = 5$

$u_4 = 8 \Rightarrow u + d = 8 \Rightarrow d = 3$   
 The numbers are  $-1, 2, 5, 8, 11$

**5**  $S_n = n(2n + 3)$   
 $S_1 = 1(2 + 3) = 5 \quad \therefore u_1 = 5$   
 $S_2 = 2(4 + 3) = 14 \quad \therefore u_1 + u_2 = 14 \quad \therefore u_2 = 9$   
 $\therefore d = 4$   
 $u_1 = 5, \quad u_2 = 9, \quad u_3 = 13, \quad u_4 = 17$

### Exercise 1D

**1 a**  $u_1 = 1 \quad r = 2 \quad u_6 = 2^5 = 32 \quad u_n = 2^{n-1}$   
**b**  $u_1 = 9 \quad r = \frac{1}{3} \quad u_6 = 9\left(\frac{1}{3}\right)^5 = \frac{1}{27} \quad u_n = 9\left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^{n-3}$   
**c**  $u_1 = x^3 \quad r = \frac{1}{x} \quad u_6 = x^3\left(\frac{1}{x}\right)^5 = \frac{1}{x^2} \quad u_n = x^3\left(\frac{1}{x}\right)^{n-1} = \left(\frac{1}{x}\right)^{n-4}$

**2 a**  $r = \frac{1}{2}, \quad u_{10} = a \quad r^9 = 48 \times \frac{1}{512} = \frac{3}{32}$   
**b**  $r = -\frac{8}{9} \div \frac{16}{3} = \frac{8}{9} \times \frac{3}{16} = -\frac{1}{6},$   
 $u_5 = a \quad r^4 = \frac{16}{3} \times \frac{1}{1296} = \frac{1}{3 \times 81} = \frac{1}{243}$

**3 a**  $a = 0.03, \quad r = 2$   
 $\Rightarrow 0.03 \times 2^{n-1} = 1.92 \Rightarrow 2^{n-1} = 64 \Rightarrow n = 7$

**b**  $a = 81, \quad r = \frac{1}{3}$   
 $81 \times \left(\frac{1}{3}\right)^{n-1} = \frac{1}{81} \Rightarrow \left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^8 \Rightarrow n = 9$

**4**  $a r^2 = 2 \quad (1)$   
 $a r^4 = 18 \quad (2)$   
 $(2) \div (1) \Rightarrow r^2 = 9 \Rightarrow r = \pm 3$   
 $u_2 = a r = \frac{2}{9} \times \pm 3 = \pm \frac{2}{3}$

**5**  $16r^4 = 9 \Rightarrow r^4 = \frac{9}{16} \Rightarrow r = \pm \frac{\sqrt{3}}{2}$   
 $\Rightarrow u_7 = 16r^6 = 16 \times \frac{27}{64} = \frac{27}{4}$

**6**  $r = \frac{a+2}{a-4} = \frac{3a+1}{a+2} \Rightarrow a^2 + 4a + 4 = 3a^2 - 11a - 4$   
 $\Rightarrow 0 = 2a^2 - 15a - 8$   
 $= (2a + 1)(a - 8)$   
 $\Rightarrow a = -\frac{1}{2} \text{ or } 8$   
 Hence  $r = \frac{\frac{1}{2}}{-4\frac{1}{2}} = -\frac{1}{3} \text{ or } r = \frac{10}{4} = \frac{5}{2}$

### Exercise 1E

**1 a**  $S_6 = \frac{2\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\frac{1}{2}} = 3.9375 \text{ or } \frac{63}{16}$   
**b**  $S_8 = \frac{2\left(1-\left(-1.5\right)^8\right)}{1-\left(-1.5\right)} = -19.7 \text{ (3 sf) or } \frac{-1261}{64}$   
**c**  $\text{Sum} = 1 + \frac{\frac{1}{2}\left(1-\left(\frac{-1}{2}\right)^9\right)}{1-\left(\frac{-1}{2}\right)} = 1.33 \text{ (3 sf) or } \frac{683}{512}$   
**d**  $u_1 = 0.1, \quad r = 0.2$   
 $\text{Sum} = \frac{0.1\left(1-0.2^{15}\right)}{1-0.2} = \frac{1}{8}(1-0.2^{15})$   
 $= \frac{1}{8}(1-\frac{1}{5^{15}})$   
 $= 0.125 \text{ (3 sf)}$

**2 a**  $\sum_{r=0}^5 5^{3-r} = 5^3 + 5^2 + 5^1 + 5^0 + 5^{-1} + 5^{-2}$   
 $= \frac{125\left(1-\left(\frac{1}{5}\right)^6\right)}{1-\frac{1}{5}}$   
 $= 156.24 \text{ or } \frac{3906}{25}$

**b**  $\sum_{r=0}^{n-1} 9 \times 10^r = 9 + 9 \times 10 + 9 \times 10^2 + \dots + 9 \times 10^{n-1}$   
 $= \frac{9(1-10^n)}{1-10}$   
 $= 10^n - 1$

**3**  $u_3 = 2 \quad u_7 = \frac{1}{128}$   
 $u_1 r^2 = 2 \quad u_1 r^6 = \frac{1}{128}$   
 $\frac{u_1 r^6}{u_1 r^2} = \frac{1}{2}$   
 $\therefore r^4 = \frac{1}{256}$   
 $r = \frac{1}{4} \text{ or } -\frac{1}{4} \quad u_1 = 32$   
 $S_6 = \frac{32\left(1-\left(\frac{1}{4}\right)^6\right)}{1-\frac{1}{4}} = \frac{1365}{32} = 42.7$   
 or  $S_6 = \frac{32\left(1-\left(-\frac{1}{4}\right)^6\right)}{1-\left(-\frac{1}{4}\right)} = \frac{819}{32} = 5.6$

**4 a**  $u_1 = S_1 = \frac{3}{2} - 1 = \frac{1}{2}$ ,  $u_2 = S_2 - S_1 = \left(\frac{3}{2}\right)^2 - \frac{3}{2} = \frac{3}{4}$ ,  
 $u_3 = S_3 - S_2 = \left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^2 = \frac{9}{8}$

**b**  $u_n = \left(\frac{3}{2}\right)^n - \left(\frac{3}{2}\right)^{n-1} = \left(\frac{3}{2}\right)^{n-1} \left(\frac{3}{2} - 1\right)$   
 $= \frac{1}{2} \times \left(\frac{3}{2}\right)^{n-1}$

This is a GP with  $u_1 = \frac{1}{2}$  and  $r = \frac{3}{2}$

**5**  $P_n = a \times ar \times ar^2 \times \dots \times ar^{n-1}$   
 $= a^n r^{1+2+\dots+n-1}$   
 $= a^n r^{\frac{(n-1)n}{2}}$

Reciprocal sequence =  $\frac{1}{a}, \frac{1}{ar}, \frac{1}{ar^2}, \dots, \frac{1}{ar^{n-1}}, \dots$

i.e. a GP with  $u_1 = \frac{1}{a}$ , and common ratio  $\frac{1}{r}$ .

$$R_n = \frac{\frac{1}{a}(1-\frac{1}{r^n})}{1-\frac{1}{r}} = \frac{1}{a} \times \frac{r^n-1}{r^n} \times \frac{r}{r-1} = \frac{1}{a} \frac{r^n-1}{(r-1)r^{n-1}}$$

$$\frac{S_n}{R_n} = \frac{a(1-r^n)}{1-r} \times \frac{a(r-1)r^{n-1}}{r^n-1} = a \times -1 \times a \times -1 r^{n-1}$$

$$= a^2 r^{n-1}$$

$$\text{Hence } \left(\frac{S_n}{R_n}\right)^n = a^{2n} r^{n(n-1)}$$

$$= \left(a^n r^{\frac{(n-1)n}{2}}\right)^2$$

$$= P_n^2 \quad \text{QED}$$

**6**  $ar = 24$

$$a r^2 = 12(P-1) \Rightarrow r = \frac{P-1}{2}$$

$$\text{But } |r| < 1 \text{ so } -1 < \frac{P-1}{2} < 1 \text{ i.e. } -2 < P-1 < 2$$

$$\Rightarrow -1 < P < 3 \quad (1)$$

$$\text{Also } S_3 = 76 \text{ so } \frac{48}{P-1} + 24 + 12(P-1) = 76$$

$$\Rightarrow 48 + 24(P-1) + 12(P-1)^2 = 76(P-1)$$

$$\Rightarrow 48 - 24 + 12P^2 + 12 = 76P - 76$$

$$\Rightarrow 12P^2 - 76P + 112 = 0$$

$$\Rightarrow 3P^2 - 19P + 28 = 0$$

$$(3P-7)(P-4) = 0$$

$$\Rightarrow P = \frac{7}{3} \text{ or } 4$$

From convergence condition (1),  $P = \frac{7}{3}$

$$\text{Hence } r = \frac{\frac{4}{3}}{\frac{2}{3}} = \frac{2}{3}$$

**7** The lengths are  $a, ar, ar^2$ ,

$$\text{Where } a + ar + ar^2 = 2 \quad (1)$$

$$\text{But } ar^2 = 2a$$

$$\text{so } r^2 = 2 \text{ and } r = \pm\sqrt{2}.$$

As  $a, ar, ar^2$  are lengths,  $r$  must be positive so  $r = \sqrt{2}$ .

$$\text{Substitute into (1)} \Rightarrow a(1 + \sqrt{2} + 2) = 2$$

$$\Rightarrow a = \frac{2}{3 + \sqrt{2}} = \frac{2}{7}(3 - \sqrt{2}) \text{ metres.}$$

**8**  $1, \frac{x+1}{3}, \frac{(x+1)^2}{9}, \frac{(x+1)^3}{27}$

Convergent when  $x = -1.5 = -\frac{3}{2}$

$$S_4 = 1 \frac{\left(1 - \frac{(x+1)^4}{3^4}\right)}{1 - \frac{x+1}{3}}$$

$$= \frac{185}{216}$$

**9**  $\frac{1}{1-r} = \frac{(1-r^n)}{1-r} = k r^{n-1}$   
 $\Rightarrow 1 - (1 - r^n) = k r^{n-1} (1 - r)$   
 $\Rightarrow r^n = k r^{n-1} (1 - r)$   
 $\Rightarrow r = k(1 - r)$   
 $\Rightarrow (1+k)r = k \Rightarrow r = \frac{k}{1+k}$   
Hence  $S = \frac{a}{1 - \frac{k}{1+k}} = \frac{a(1+k)}{1+k-k}$   
 $= a(1+k) = (k+1)a$   
 $= (k+1)u_1$

### Exercise 1F

**1 a**  $S = 4 u_2 \frac{u_1}{1-r} = 4 u_1 r$

$$1 = 4r(1-r)$$

$$1 = 4r - 4r^2$$

$$4r^2 - 4r + 1 = 0$$

$$(2r-1)^2 = 0$$

$$r = \frac{1}{2}$$

**b**  $u_1 = 32 \quad r = \frac{1}{2} \quad S_5 = \frac{32 \left(1 - \left(\frac{1}{2}\right)^5\right)}{1 - \frac{1}{2}} = 62$

$$S = \frac{32}{1 - \frac{1}{2}} = 64$$

$$\text{percentage error} = \frac{2}{62} \times 100 = 3.23\%$$

**2**  $r = 1.5 \quad S_5 = 52750$

$$\frac{u_1(1-1.5^5)}{1-1.5} = 52750$$

$$u_1 = \$4000$$

**3 a**  $2 + 4 + 8 + 16 + 32 = 62$

**b**  $S_n > 1000000$

$$\frac{2(1-2^n)}{1-2} > 1000000$$

$$(2^n - 1) > 500000$$

$$2^n > 500001$$

$$n = 19$$

- 4 a** Let  $x$  = monthly repayment

$$\begin{aligned}\text{Amount owing after 1 month} \\ = 1000 \times 1.01 - x\end{aligned}$$

Amount owing after 2 months

$$\begin{aligned} &= (1000 \times 1.01 - x) \times 1.01 - x \\ &= 1000 \times 1.01^2 - 1.01x - x\end{aligned}$$

Amount owing after 3 months

$$\begin{aligned} &= (1000 \times 1.01^2 - 1.01x - x) \times 1.01 - x \\ &= 1000 \times 1.01^3 - 1.01^2x - 1.01x - x\end{aligned}$$

Amount owing after 24 months

$$\begin{aligned} &= 1000 \times 1.01^{24} - 1.01^{23}x - 1.01^{22}x \\ &\quad - 1.01^{21}x \dots - 1.01x - x\end{aligned}$$

We require this to be zero

$$\begin{aligned}\therefore x + 1.01x + 1.01^2x + \dots + 1.01^{23}x \\ = 1000 \times 1.01^{24}\end{aligned}$$

$$\frac{x(1-1.01^{24})}{1-1.01} = 1000 \times 1.01^{24}$$

$$x = \$47.07$$

- b** Total to be paid =  $47.07 \times 24$   
= \$1130

## Exercise 1G

- 1 a** Odd number + even number =  $2a + 1 + 2b$   
=  $2(a + b) + 1$ ,

which is odd.

- b** Odd number  $\times$  odd number =  $(2m + 1)(2n + 1)$   
=  $4mn + 2m + 2n + 1 = 2(m + n + 2mn) + 1$ ,  
which is odd.

$$\begin{aligned}2 \quad \frac{1}{x-2} - \frac{2}{2x+5} &= \frac{2x+5-2(x-2)}{(x-2)(2x+5)} \\ \therefore \frac{1}{x-2} - \frac{2}{2x+5} &= \frac{9}{2x^2+x-10}\end{aligned}$$

$$3 \quad (a+b)^2 = c^2 + 4\left(\frac{ab}{2}\right)$$

$$a^2 + 2ab + b^2 = c^2 + 2ab$$

$$\therefore a^2 + b^2 = c^2$$

3	4	$3 \times 4 + 4$	16
7	8	$7 \times 8 + 8$	64
-6	-5	$-6 \times -5 + -5$	25
11	12	$11 \times 12 + 12$	144
8	9	$8 \times 9 + 9$	81

The product of two consecutive integers plus the larger of the two integers is equal to the square of the larger integer.

Proof: Let the two integers be  $n$  and  $n + 1$

$$n(n+1) + (n+1) = (n+1)(n+1) = (n+1)^2$$

## Exercise 1H

$$1 \quad p(n): S_n = \frac{u_1(1-r^n)}{1-r}$$

Step 1: when  $n = 1$ , LHS =  $S_1 = u_1$

$$\text{RHS} = \frac{u_1(1-r)}{1-r} = u_1$$

$\therefore p(1)$  is true

$$\text{Step 2: assume } p(k) \text{ i.e., } S_k = \frac{u_1(1-r^k)}{1-r}$$

$$\text{Step 3: prove } p(k+1) \text{ i.e., } S_{k+1} = \frac{u_1(1-r^{k+1})}{1-r}$$

$$\begin{aligned}\text{Proof: } S_{k+1} &= S_k + u_{k+1} \\ &= S_k + u_1 r^k \\ &= \frac{u_1(1-r^k)}{1-r} + u_1 r^k \\ &= \frac{u_1(1-r^k) + u_1 r^k (1-r)}{1-r} \\ &= \frac{u_1(1-r^k + r^k - r^{k+1})}{1-r} \\ &\therefore S_{k+1} = \frac{u_1(1-r^{k+1})}{1-r}\end{aligned}$$

Since  $p(1)$  is true and if  $p(k)$  is true then  $p(k+1)$  is true, by the principle of mathematical induction,  $p(n)$  is true

$$2 \quad \text{a} \quad p(n): \sum_{r=1}^n r^2 = \frac{n}{6}(n+1)(2n+1)$$

Step 1: when  $n = 1$ , LHS = 1

$$\text{RHS} = \frac{1}{6}(2)(3) = 1$$

$\therefore p(1)$  is true

$$\text{Step 2: assume } p(k) \text{ i.e., } \sum_{r=1}^k r^2 = \frac{k}{6}(k+1)(2k+1)$$

$$\text{Step 3: prove } p(k+1) \text{ i.e., } \sum_{r=1}^{k+1} r^2 = \frac{(k+1)}{6}(k+2)(2k+3)$$

$$\begin{aligned}\text{Proof: } \sum_{r=1}^{k+1} r^2 &= \sum_{r=1}^k r^2 + (k+1)^2 \\ &= \frac{k}{6}(k+1)(2k+1) + (k+1)^2 \\ &= \frac{(k+1)}{6}[k(2k+1) + 6(k+1)] \\ &= \frac{(k+1)}{6}[2k^2 + 7k + 6] \\ \therefore \sum_{r=1}^{k+1} r^2 &= \frac{(k+1)}{6}(k+2)(2k+3)\end{aligned}$$

Since  $p(1)$  is true and if  $p(k)$  is true then  $p(k+1)$  is true, by the principle of mathematical induction,  $p(n)$  is true.

$$b \quad p(n): \sum_{r=1}^n 2^{r-1} = 2^n - 1$$

Step 1: when  $n = 1$ , LHS =  $2^0 = 1$

$$\text{RHS} = 2^1 - 1 = 1$$

$\therefore p(1)$  is true

$$\text{Step 2: assume } p(k) \text{ i.e., } \sum_{r=1}^k 2^{r-1} = 2k - 1$$

Step 3: prove  $p(k+1)$  i.e.,  $\sum_{r=1}^{k+1} 2^{r-1} = 2^{k+1} - 1$

$$\begin{aligned}\text{Proof: } \sum_{r=1}^{k+1} 2^{r-1} &= \sum_{r=1}^k 2^{r-1} + 2^k \\ &= 2^k - 1 + 2^k = 2(2^k) - 1 \\ \therefore \sum_{r=1}^{k+1} 2^{r-1} &= 2^{k+1} - 1\end{aligned}$$

Since  $p(1)$  is true and if  $p(k)$  is true then  $p(k+1)$  is true, by the principle of mathematical induction,  $p(n)$  is true.

c  $p(n)$ :  $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2}{4}(n+1)^2$

Step 1: when  $n = 1$ , LHS =  $1^3 = 1$

$$\text{RHS} = \frac{1}{4}(2)^2 = 1$$

$\therefore p(1)$  is true

Step 2: assume  $p(k)$  i.e.,  $1^3 + 2^3 + 3^3 + \dots + k^3 = \frac{k^2}{4}(k+1)^2$

$$= \frac{k^2}{4}(k+1)^2$$

Step 3: prove  $p(k+1)$  i.e.,

$$\begin{aligned}1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 \\ = \frac{(k+1)^2}{4}(k+2)^2\end{aligned}$$

Proof:  $1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3$

$$= \frac{k^2}{4}(k+1)^2 + (k+1)^3 = \frac{(k+1)^2}{4}[k^2 + 4k + 4]$$

$$\therefore 1^3 + 2^3 + 3^3 + \dots + (k+1)^3 = \frac{(k+1)^2}{4}(k+2)^2$$

Since  $p(1)$  is true and if  $p(k)$  is true then  $p(k+1)$  is true, by the principle of mathematical induction,  $p(n)$  is true.

d  $p(n)$ :  $\sum_{r=1}^n r(r+2) = \frac{n}{6}(n+1)(2n+7)$

Step 1: when  $n = 1$ , LHS =  $1(3) = 3$

$$\text{RHS} = \frac{1}{6}(2)(9) = 3$$

$\therefore p(1)$  is true

Step 2: assume  $p(k)$  i.e.,

$$\sum_{r=1}^k r(r+2) = \frac{k}{6}(k+1)(2k+7)$$

Step 3: prove  $p(k+1)$  i.e.,

$$\sum_{r=1}^{k+1} r(r+2) = \frac{(k+1)}{6}(k+2)(2k+9)$$

$$\begin{aligned}\text{Proof: } \sum_{r=1}^{k+1} r(r+2) &= \sum_{r=1}^k r(r+2) + (k+1)(k+3) \\ &= \frac{k}{6}(k+1)(2k+7) + (k+1)(k+3) \\ &= \frac{(k+1)}{6}[k(2k+7) + 6(k+3)] \\ &= \frac{(k+1)}{6}(2k^2 + 13k + 18)\end{aligned}$$

$$\therefore \sum_{r=1}^{k+1} r(r+2) = \frac{(k+1)}{6}(k+2)(2k+9)$$

Since  $p(1)$  is true and if  $p(k)$  is true then  $p(k+1)$  is true, by the principle of mathematical induction,  $p(n)$  is true.

## Exercise 11

1  $p(n)$ :  $7^n - 1 = 6A$  ( $A \in \mathbb{Z}$ )

Step 1: when  $n = 1$ ,  $7^n - 1 = 7 - 1 = 6$

$\therefore p(1)$  is true

Step 2: assume  $p(k)$  i.e.,

$$7^k - 1 = 6A$$

Step 3: prove  $p(k+1)$  i.e.,

$$7^{k+1} - 1 = 6B$$
 ( $B \in \mathbb{Z}$ )

$$\text{Proof: } 7^{k+1} - 1 = 7(7^k) - 1$$

$$= 7(6A + 1) - 1$$

$$= 42A + 7 - 1$$

$$= 42A + 6$$

$$= 6(7A + 1)$$

$$\therefore 7^{k+1} - 1 = 6B$$

Since  $p(1)$  is true and if  $p(k)$  is true then  $p(k+1)$  is true, by the principle of mathematical induction,  $p(n)$  is true.

2  $p(n)$ :  $1 + 3 + 5 + 7 + \dots + (2n-1) = n^2$

Step 1: when  $n = 1$ , LHS = 1

$$\text{RHS} = 1^2 = 1$$

$\therefore p(1)$  is true

Step 2: assume  $p(k)$  i.e.,

$$1 + 3 + 5 + 7 + \dots + (2k-1) = k^2$$

Step 3: prove  $p(k+1)$  i.e.,

$$\begin{aligned}1 + 3 + 5 + 7 + \dots + (2k-1) + (2k+1) \\ = (k+1)^2\end{aligned}$$

$$\text{Proof: } 1 + 3 + 5 + 7 + \dots + (2k-1) + (2k+1) \\ = k^2 + (2k+1)$$

$$\therefore 1 + 3 + 5 + 7 + \dots + (2k+1) = (k+1)^2$$

Since  $p(1)$  is true and if  $p(k)$  is true then  $p(k+1)$  is true, by the principle of mathematical induction,  $p(n)$  is true.

3  $p(n)$ :  $9^n - 1 = 8A$ , where  $A \in \mathbb{Z}$

Step 1: when  $n = 1$ ,  $9^n - 1 = 8$   $p(1)$  is true

Step 2: Assume  $p(k)$  i.e.

$$9^k - 1 = 8A$$

Step 3: prove  $p(k+1)$  i.e.

$$9^{k+1} - 1 = 8B$$
 ( $B \in \mathbb{Z}$ )

$$\text{Proof: } 9^{k+1} - 1 = 9 \times 9^k - 1$$

$$= 9(8A + 1) - 1$$

$$= 72A + 8$$

$$= 8(9A + 1)$$

$$\therefore 9^{k+1} - 1 = 8B$$

Since  $p(1)$  is true and if  $p(k)$  is true then  $p(k+1)$  is true, by induction,  $p(n)$  is true

**4**  $p(n)$ :  $n^3 - n = 6A$ , where  $A \in \mathbb{Z}$

Step 1: when  $n = 1$ ,  $n^3 - n = 0 = 6 \times 0$   
 $\therefore p(1)$  is true

Step 2: Assume  $p(k)$  i.e.

$$k^3 - k = 6A$$

Step 3: prove  $p(k+1)$  i.e.  $(k+1)^3 - (k+1) = 6B$   
where  $B \in \mathbb{Z}$

Proof:  $(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1$   
 $= k^3 + 3k^2 + 2k$   
 $= 6A + k + 3k^2 + 2k$   
 $= 6A + 3(k^2 + k)$   
 $= 6A + 3k(k+1)$

But  $k(k+1)$  is either odd  $\times$  even or even  $\times$  odd so  
is divisible by 2.

$\therefore 3k(k+1)$  is divisible by 6.

$$\therefore (k+1)^3 - (k+1) = 6B$$

$\therefore$  Since  $p(1)$  is true and if  $p(k)$  is true then  
 $p(k+1)$  is true, by induction  $p(n)$  is true.

**5**  $p(n)$ :  $\sum_{r=1}^n \frac{1}{r(r+1)} = \frac{n}{n+1}$

Step 1: when  $n = 1$ ,  $\sum_{r=1}^n \frac{1}{r(r+1)} = \frac{1}{1 \times 2} = \frac{1}{2}$

$$\text{and } \frac{n}{n+1} = \frac{1}{2} \therefore p(1) \text{ is true}$$

Step 2: assume  $p(k)$  i.e.  $\sum_{r=1}^k \frac{1}{r(r+1)} = \frac{k}{k+1}$

Step 3: prove  $p(k+1)$  i.e.  $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k+1}{k+2}$

Proof:  $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^k \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)}$   
 $= \frac{k^2 + 2k + 1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$

Since  $p(1)$  is true, and if  $p(k)$  is true then  $p(k+1)$  is true, by induction  $p(n)$  is true.

**6**  $p(n)$ :  $2^{n+2} + 3^{2n+1} = 7A$  where  $A \in \mathbb{Z}$

Step 1: when  $n = 1$ ,  $2^{n+2} + 3^{2n+1} = 2^2 + 3^3$   
 $= 8 + 27 = 35 + 7 \times 5$

$$\therefore p(1) \text{ is true}$$

Step 2: assume  $p(k)$  i.e.  $2^{k+2} + 3^{2k+1} = 7A$

Step 3: prove  $p(k+1)$  i.e.  $2^{k+3} + 3^{2k+3} = 7B$   
where  $B \in \mathbb{Z}$

Proof:  $2^{k+3} + 3^{2k+3} = 2(7A - 3^{2k+1}) + 3^{2k+3}$   
 $= 14A + 3^{2k+3} - 2 \times 3^{2k+1}$   
 $= 14A + 3^{2k+1}(9 - 2)$   
 $= 14A + 3^{2k+1} \times 7$   
 $= 7(2A + 3^{2k+1}) = 7B$

Since  $p(1)$  is true, and if  $p(k)$  is true then  $p(k+1)$  is true, by induction,  $p(n)$  is true.

**7**  $1, \frac{1}{3}, -\frac{1}{9}, -\frac{11}{27}, -\frac{49}{81}$

$$p(n): u_n = 3\left(\frac{2}{3}\right)^n - 1$$

Step 1: When  $n = 1$   $u_1 = 1$  and  $3\left(\frac{2}{3}\right)^1 - 1 = 2 - 1 = 1$   
 $\therefore p(1)$  is true

Step 2: Assume  $p(k)$  i.e.  $u_k = 3\left(\frac{2}{3}\right)^k - 1$

Step 3: Prove  $p(k+1)$  i.e.  $u_{k+1} = 3\left(\frac{2}{3}\right)^{k+1} - 1$

Proof:  $u_{k+1} = \frac{2u_k - 1}{3}$

$$= \frac{2 \times 3\left(\frac{2}{3}\right)^k - 2 - 1}{3}$$

$$= \frac{2 \times 3\left(\frac{2}{3}\right)^k - 3}{3}$$

$$= 2 \times \left(\frac{2}{3}\right)^k - 1$$

$$= \frac{2}{3} \times 3 \times \left(\frac{2}{3}\right)^k - 1$$

$$= 3\left(\frac{2}{3}\right)^{k+1} - 1$$

Since  $p(1)$  is true, and if  $p(k)$  is true then  $p(k+1)$  is true, therefore by induction,  $p(n)$  is true.

## Exercise 1J

**1**  $8! - 7! = 8 \times 7! - 7! = 7 \times 7!$

$$10! - 9! = 10 \times 9! - 9! = 9 \times 9!$$

$$5! - 4! = 5 \times 4! - 4! = 4 \times 4!$$

$$95! - 94! = 95 \times 94! - 94! = 94 \times 94!$$

$$(n+1)! - n! = (n+1)n! - n! = n \times n!$$

**2** **a**  $\frac{4!}{6!} = \frac{1}{6 \times 5} = \frac{1}{30}$

**b**  $\frac{5! \times 3!}{6!} = \frac{3!}{6} = 1$

**c**  $\frac{8! \times 6!}{5!} = 8! \times 6 = 241920$

**3** **a**  $\frac{n! + (n-1)!}{(n+1)!} = \frac{n(n-1)! + (n-1)!}{(n+1)n(n-1)!} = \frac{n+1}{(n+1)n} = \frac{1}{n}$

**b**  $\frac{n! - (n-1)!}{(n-2)!} = \frac{n(n-1)(n-2)! - (n-1)(n-2)!}{(n-2)!}$

$$= n(n-1) - (n-1)$$

$$= (n-1)(n-1)$$

$$= (n-1)^2$$

**c**  $\frac{(n!)^2 - 1}{n! + 1} = \frac{(n!-1)(n!+1)}{n!+1} = n! - 1$

**4**  $\frac{(2n+2)!(n!)^2}{[(n+1)!]^2 (2n)!} = \frac{(2n+2)(2n+1)(2n)!(n!)^2}{(n+1)^2 (n!)^2 (2n)!}$   
 $= \frac{2(n+1)(2n+1)}{(n+1)^2}$   
 $= \frac{2(2n+1)}{(n+1)}$

**Exercise 1K**

**1**  $26 \times 25 \times 24 = 15600$

**2 a**  $12! = 479\,001\,600$

**b**  $4! \times 3! \times 4! \times 2! \times 3! = 41\,472$

**3**  $\binom{8}{4} = 70$  weeks

**4 a**  $\binom{20}{4} = 4845$

**b**  $4845 - \binom{8}{4} - \binom{12}{4} = 4845 - 70 - 495 = 4280$

**5 a**  $6 \times 7 \times 7 \times 4 = 1176$

**b** must end in 0  $6 \times 7 \times 7 \times 1 = 294$

**c** ending in 0  $6 \times 5 \times 4 \times 1 = 120$

ending in 2, 4 or 6  $5 \times 5 \times 4 \times 3 = 300$

$120 + 300 = 420$

**6**  $26^3 \times 10^3 = 17\,576\,000$

**Exercise 1L**

**1 a**  $\binom{n}{r} = \frac{n!}{(n-r)!r!}$   $\binom{n}{n-r} = \frac{n!}{(n-(n-r))!(n-r)!} = \frac{n!}{r!(n-r)!}$   
 $\therefore \binom{n}{r} = \binom{n}{n-r}$

**b**  $\binom{n+1}{r} = \frac{(n+1)!}{(n+1-r)!r!}$

$$\begin{aligned}\binom{n}{r} + \binom{n}{r-1} &= \frac{n!}{(n-r)!r!} + \frac{n!}{(n-r+1)!(r-1)!} \\ &= \frac{(n-r+1)n! + rn!}{(n-r+1)!r!}\end{aligned}$$

$$= \frac{n \times n! + n!}{(n-r+1)!r!}$$

$$= \frac{n!(n+1)}{(n-r+1)!r!}$$

$$= \frac{(n+1)!}{(n-r+1)!r!}$$

$$\therefore \binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

**2 a**  $(1+2x)^{11} = 1 + \binom{11}{1}(2x) + \binom{11}{2}(2x)^2 + \binom{11}{3}(2x)^3 + \dots$   
 $= 1 + 22x + 220x^2 + 1320x^3 + \dots$

**b**  $(1-3x)^7 = 1 + \binom{7}{1}(-3x) + \binom{7}{2}(-3x)^2 + \binom{7}{3}(-3x)^3 + \dots$   
 $= 1 - 21x + 189x^2 - 945x^3 + \dots$

**c**  $(2+5x)^5 = 2^5 + \binom{5}{1}2^4(5x) + \binom{5}{2}2^3(5x)^2 + \binom{5}{3}2^2(5x)^3 + \dots$

$$= 32 + 400x + 2000x^2 + 5000x^3 + \dots$$

**d**  $\left(2-\frac{x}{3}\right)^9 = 2^9 + \binom{9}{1}2^8\left(\frac{-x}{3}\right) + \binom{9}{2}2^7\left(\frac{-x}{3}\right)^2 + \binom{9}{3}2^6\left(\frac{-x}{3}\right)^3 + \dots$   
 $= 512 - 768x + 512x^2 - \frac{1792}{9}x^3 + \dots$

**3 a**  $(1-4x)^7$  4th term  $= \binom{7}{3}(-4x)^3 = -2240x^3$

**b**  $\left(1-\frac{x}{2}\right)^{20}$  3rd term  $= \binom{20}{2}\left(\frac{-x}{2}\right)^2 = \frac{95}{2}x^2$

**c**  $(2a-b)^8$  4th term  $= \binom{8}{3}(2a)^5(-b)^3 = -1792a^5b^3$

**4**  $\binom{12}{4}(2x)^8\left(\frac{1}{x^2}\right)^4 = 126720$

**5**  $(2+\frac{x}{5})^5 = 2^5 + \binom{5}{1}2^4 \cdot \frac{x}{5} + \binom{5}{2}2^3 \cdot \frac{x^2}{25} + \binom{5}{3}2^2 \cdot \frac{x^3}{125}$   
 $+ \binom{5}{4}2 \cdot \frac{x^4}{625} + \frac{x^5}{3125}$

$$= 32 + 16x + \frac{80x^2}{25} + \frac{40x^3}{125} + \frac{2x^4}{125} + \frac{x^5}{3125}$$

$$(2.01)^5 = (2 + \frac{0.05}{5})^5 = 32 + 0.8 + 0.008 + 0.00004 + 0.0000001 + \dots$$

$$= 32.80804 \text{ (5 dp)}$$

**6 a**  $(\sqrt{2}-\sqrt{3})^4 = 4 - 4 \times 2\sqrt{2} \times \sqrt{3} + 6 \times 2 \times 3 - 4 \times \sqrt{2} \times 3\sqrt{3}$   
 $= 4 - 8\sqrt{6} + 36 - 12\sqrt{6} + 9$   
 $= 49 - 20\sqrt{6}$

**b**  $(\sqrt{2} + \frac{1}{\sqrt{5}})^3 = 2\sqrt{2} + 3 \times 2 \times \frac{1}{\sqrt{5}}$   
 $+ 3\sqrt{2} \times \frac{1}{\sqrt{5}} + \frac{1}{5\sqrt{5}}$   
 $= \frac{13\sqrt{2}}{5} + \frac{31}{5\sqrt{5}} = \frac{13}{5}\sqrt{2} + \frac{31}{5}\sqrt{5}$

**c**  $(1+\sqrt{7})^5 - (1-\sqrt{7})^5 = 2 \times 5 \times \sqrt{7} + 2 \times 10 \times (\sqrt{7})^3 + 2 \times (\sqrt{7})^5$   
 $= 10\sqrt{7} + 140\sqrt{7} + 98\sqrt{7}$   
 $= 248\sqrt{7}$

**7 a**  $a^2 - b^2 = x^2 + 2xy + y^2 - (x^2 - 2xy + y^2)$   
 $= 4xy$   
 $= 4\frac{(a+b)}{2}, \frac{(a-b)}{2}$  (using  $2x = a + b$  and  
 $2y = a - b$ )  
 $= (a+b)(a-b) = (a-b)(a+b)$

**b** 
$$\begin{aligned} a^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\ b^3 &= x^3 - 3x^2y + 3xy^2 - y^3 \Rightarrow a^3 - b^3 = 6x^2y + 2y^3 \\ &= 2y(3x^2 + y^2) = (a - b)(3x^2 + y^2) \\ &= (a - b) \left[ \frac{3(a+b)^2}{2^2} + \left( \frac{a-b}{2} \right)^2 \right] \\ &= \frac{(a-b)}{4} [3a^2 + 6ab + 3b^2 + a^2 - 2ab + b^2] \\ &= \frac{(a-b)}{4} (4a^2 + 4ab + 4b^2) \\ &= (a - b)(a^2 + ab + b^2) \end{aligned}$$

**c** 
$$\begin{aligned} a^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\ b^4 &= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 \\ \Rightarrow a^4 - b^4 &= 8x^3y + 8xy^3 = 8xy(x^2 + y^2) \\ &= 8 \frac{(a+b)}{2} \frac{(a-b)}{2} (x^2 + y^2) \\ &= 2(a-b)(a-b) \left[ \left( \frac{a+b}{2} \right)^2 + \left( \frac{a-b}{2} \right)^2 \right] \\ &= 2(a-b)(a+b) \left[ \frac{a^2}{2} + \frac{b^2}{2} \right] \\ &= (a-b)(a+b)(a^2 + b^2) \end{aligned}$$

**d**  $(a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + b^{n-1})$

**e** Let  $p(n)$  be  $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \dots + b^{n-1})$

When  $n = 1$ ,  $a^1 - b^1 = a - b$  so  $p(1)$  is true.

Assume  $p(n)$  is true for  $n = k$  i.e.  $a^k - b^k$

$= (a-b)(a^{k-1} + a^{k-2}b + \dots + b)$

Prove  $p(n)$  is true for  $n = k+1$ :

$$\begin{aligned} a^{k+1} - b^{k+1} &= a \times a^k - b^{k+1} \\ &= a [(a-b)(a^{k-1} + a^{k-2}b + \dots + b^{k-1})] + a \times b^k - b^{k+1} \\ &= (a-b)[a^k + a^{k-1}b + \dots + ab^{k-1}] + ab^k - b^{k+1} \\ &= (a-b)(a^k + a^{k-1}b + \dots + ab^{k-1}) + (a-b)b^k + b^{k+1} - b^{k+1} \\ &= (a-b)(a^k + a^{k-1}b + \dots + ab^{k-1} + b^k) \end{aligned}$$

$\therefore p(k+1)$  is true.

So, since  $p(1)$  is true and if  $p(k)$  is true then  $p(k+1)$  is true, therefore by induction  $p(n)$  is true.



## Review exercise

- 1**  $u_2 = 16$   $S_3 = 84$   
 $u_1 r = 16$   $u_1 + u_1 r + u_1 r^2 = 84$   
 $u_1 = \frac{16}{r}$   $u_1(1 + r + r^2) = 84$   
 $\frac{16}{r}(1 + r + r^2) = 84$   
 $16 + 16r + 16r^2 = 84r$   
 $16r^2 - 68r + 16 = 0$   
 $4r^2 - 17r + 4 = 0$   
 $(4r-1)(r-4) = 0$   $r = \frac{1}{4}$  or  $4$   
if  $r = \frac{1}{4}$ ,  $u_1 = 64$   $64, 16, 4$   
if  $r = 4$ ,  $u_1 = 4$   $4, 16, 64$

**2** 
$$\begin{aligned} 1 + 3 + 4 + 6 + 7 + 9 + 10 + 12 + \dots + 46 \\ &= (1 + 4 + 7 + \dots + 46) + (3 + 6 + 9 + \dots + 45) \\ &= \frac{16}{2}(1 + 46) + \frac{15}{2}(3 + 45) \\ &= 376 + 360 = 736 \end{aligned}$$

**3**  $c - b = b - a$   $\frac{b}{a} = \frac{a}{c}$   $a + b + c = \frac{-9}{2}$  (3)  
 $\therefore a + c = 2b$  (1)  $\therefore bc = a^2$  (2)  
substitute (1) in (3)  $2b + b = \frac{-9}{2}$   
 $3b = \frac{-9}{2}$   $\therefore b = \frac{-3}{2}$   
 $a + c = -3$   $\frac{-3}{2}c = a^2$   
 $c = -3 - a$   $\therefore \frac{-3}{2}(-3 - a) = a^2$   
 $9 + 3a = 2a^2$   
 $2a^2 - 3a - 9 = 0$   
 $(2a+3)(a-3) = 0$   
 $a = \frac{-3}{2}$  or  $3$   
 $a \neq \frac{-3}{2}$  since  $a \neq b$   $\therefore a = 3$ ,  $c = -6$

The three numbers are  $3, \frac{-3}{2}, -6$

**4**  $1, 3, 7, 15, 31, 63$

$p(n): u_n = 2^n - 1$

Step 1: when  $n = 1$ ,  $u_1 = 1 = 2^1 - 1$

$\therefore p(1)$  is true.

Step 2: assume  $p(k)$  i.e.  $u_k = 2^k - 1$

Step 3: prove  $p(k+1)$  i.e.  $u_{k+1} = 2^{k+1} - 1$

$$\begin{aligned} \text{proof: } u_{k+1} &= 2u_k + 1 \\ &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

Since  $p(1)$  is true and if  $p(k)$  is true then  $p(k+1)$  is true, by the principal of mathematical induction,  $p(n)$  is true.

**5**  $p(n): 3^{2n} - 8n - 1 = 64A$  ( $A \in \mathbb{Z}, \in \mathbb{Z}^+$ )

Step 1: when  $n = 1$ ,  $3^2 - 8 - 1 = 0$

$\therefore p(1)$  is true.

Step 2: assume  $p(k)$  i.e.  $3^{2k} - 8k - 1 = 64A$

Step 3: prove  $p(k+1)$  i.e.  $3^{2(k+1)} - 8(k+1) - 1 = 64B$  ( $B \in \mathbb{Z}$ )

Proof:  $3^{2(k+1)} - 8(k+1) - 1$

$$\begin{aligned} &= 3^{2k}(3^2) - 8k - 9 \\ &= 9(64A + 8k + 1) - 8k - 9 \\ &= 576A + 72k + 9 - 8k - 9 \\ &= 576A + 64k \\ &= 64(9A + k) \\ &= 6B \end{aligned}$$

Since  $p(1)$  is true and if  $p(k)$  is true then  $p(k+1)$  is true, by the principal of mathematical induction,  $p(n)$  is true.

**6 a**  $\binom{n+1}{4} = \frac{(n+1)!}{(n-3)!4!}$       **b**  $\binom{n-1}{2} = \frac{(n-1)!}{(n-3)!2!}$

**c**  $\frac{(n+1)!}{(n-3)!4!} = \frac{6(n-1)!}{(n-3)!2!}$

$$\frac{(n+1)n}{24} = 3$$

$$n^2 + n = 72$$

$$n^2 + n - 72 = 0$$

$$(n+9)(n-8) = 0$$

$\therefore n = 8$  ( $n$  cannot be negative)

**7**  $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{r}x^r + \dots + \binom{n}{n}x^n$

**a** Let  $x = 1$ ,  $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{r} + \dots + \binom{n}{n} = 2^n$

**b** Let  $x = -1$ ,  $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^r \binom{n}{r} + \dots + (-1)^n \binom{n}{n} = 0$

**2 a**  $\frac{14!}{3!2!2!2!2!} = 908\,107\,200$

**b** Consider 5 digit and 6 digit numbers ending in 0 or 5.

5 digit numbers:

$$4 \times 6 \times 6 \times 6 \times 2 = 1728$$

6 digit numbers:

$$5 \times 6 \times 6 \times 6 \times 6 \times 2 = 12960$$

$$1728 + 12960 = 14688$$

**c**  $4! \times (2!)^4 = 384$

**3**

<b>M</b>	<b>W</b>
2	3
1	4

$$\begin{aligned} &\binom{6}{2} \times \binom{4}{3} + \binom{6}{1} \times \binom{4}{4} \\ &= 15 \times 4 + 6 \times 1 \\ &= 66 \end{aligned}$$

**4**  $\binom{8}{6} = (x^3)^2 \left(-\frac{3}{x}\right)^6 = 20412$

**5** Coefficients are  $\binom{n}{r-1}, \binom{n}{r}, \binom{n}{r+1}$

$$\frac{n!}{(n-r-1)!(r-1)!} - \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)!r!} - \frac{n!}{(n-r+1)!(r-1)!}$$

Divide by  $n!$  and multiply by  $(r+1)!(n-r+1)!$

$$(n-r+1)(n-r) - (r+1)$$

$$(n-r+1) = (r+1)$$

$$(n-r+1) - (r+1)r$$

$$(n-r+1)(n-r) - 2$$

$$(r+1)(n-r+1) +$$

$$(r+1)r = 0$$

$$n^2 - rn - rn + r^2 + n - r - 2rn + 2r^2 - 2r - 2n + 2r - 2 + r^2 + r = 0$$

$$n^2 - 4rn + 4r^2 - n - 2 = 0$$

$$n^2 + 4r^2 - 2 - n(4r+1) = 0$$

$$n = 14, \quad 196 + 4r^2 - 2 - 14(4r+1) = 0$$

$$4r^2 - 56r + 180 = 0$$

$$r^2 - 14r + 45 = 0$$

$$(r-5)(r-9) = 0$$

$$r = 5 \text{ or } 9$$

The coefficients are  $\binom{14}{4}, \binom{14}{5}, \binom{14}{6}$  or  $\binom{14}{8}, \binom{14}{9}, \binom{14}{10}$

Both sets give 1001, 2002, 3003.



## Review exercise

**1 a**  $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = a^2$        $a^2 = \frac{1}{2}$        $a = \frac{1}{\sqrt{2}}$

$$\left(\frac{1}{2\sqrt{2}}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2 = b^2$$

$$b^2 = \frac{1}{4}$$

$$b = \frac{1}{2}$$

$$\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = c^2$$

$$c^2 = \frac{1}{8}$$

$$c = \frac{1}{2\sqrt{2}}$$

**b** The spiral consists of 1.5 of the sides of the first eight squares and one of the sides of the ninth square.

$$\begin{aligned} \text{length} &= 1.5 \left(1 + \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \dots + \left(\frac{1}{\sqrt{2}}\right)^7\right) + \left(\frac{1}{\sqrt{2}}\right)^8 \\ &= 1.5 \left(\frac{1 - \left(\frac{1}{\sqrt{2}}\right)^8}{1 - \frac{1}{\sqrt{2}}}\right) + \frac{1}{16} = 4.86 \end{aligned}$$

**c** length =  $1.5 \left(\frac{1}{1 - \frac{1}{\sqrt{2}}}\right) = 5.12$

**d** The spiral consists of 8 triangles

$$\text{Area} = \frac{1}{2} \left( \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2 + \left(\frac{1}{4}\right)^2 + \dots \right) \text{ to 8 terms}$$

$$= \frac{1}{2} \left( \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \right) \text{ to 8 terms}$$

$$\text{Area} = \frac{1}{2} \left( \frac{\frac{1}{4} \left(1 - \left(\frac{1}{2}\right)^8\right)}{1 - \frac{1}{2}} \right) = 0.249$$

**e** Area =  $\frac{1}{2} \left( \frac{\frac{1}{4}}{1 - \frac{1}{2}} \right) = 0.25$