

# 11

# Inspiration and formalism

## Answers

### Skills check

1 P(-1, 5) Q(2, 1)  $PQ^2 = (-1 - 2)^2 + (5 - 1)^2 = 25$

$PQ = 5$

2 A(1, 3) B(4, 9)

a gradient =  $\frac{9 - 3}{4 - 1} = 2$

b  $y - 3 = 2(x - 1) \Rightarrow y = 2x + 1$

### Exercise 11A

1 a opposite sides of a regular hexagon are equal and parallel  $\therefore \overrightarrow{AB} = \overrightarrow{ED}$

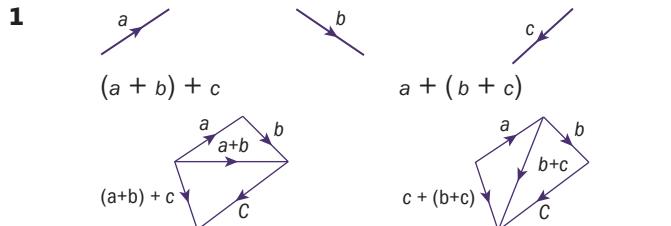
b i  $\overrightarrow{FC}$  and  $\overrightarrow{ED}$

ii  $\overrightarrow{AD}, \overrightarrow{DA}, \overrightarrow{BE}, \overrightarrow{EB}$  and  $\overrightarrow{FC}$

2 a No, 2 sides of a pentagon are parallel  $\therefore$  the vectors in the diagram are all distinct.

b No, since no 2 vectors in the diagram are parallel and equal in length

### Exercise 11B



2 a  $\overrightarrow{AF} + \overrightarrow{BC} = \overrightarrow{AF} + \overrightarrow{FE} = \overrightarrow{AE}$

b  $\frac{1}{2}\overrightarrow{AD} + \overrightarrow{ED} = \overrightarrow{FE} + \overrightarrow{ED} = \overrightarrow{FD}$

c  $2\overrightarrow{FE} - \overrightarrow{AF} - \overrightarrow{FE} = \overrightarrow{FE} + \overrightarrow{FA}$   
 $= \overrightarrow{FE} + \overrightarrow{EO} = \overrightarrow{FO} = \overrightarrow{AB}$

d  $\frac{1}{2}(\overrightarrow{AD} + \overrightarrow{BE}) = \frac{1}{2}\overrightarrow{AD} + \frac{1}{2}\overrightarrow{BE} = \overrightarrow{AO} + \overrightarrow{OE} = \overrightarrow{AE}$

e  $-\frac{1}{2}\overrightarrow{FC} + \overrightarrow{BC} = \overrightarrow{OF} + \overrightarrow{FE} = \overrightarrow{OE} = \overrightarrow{CD}$

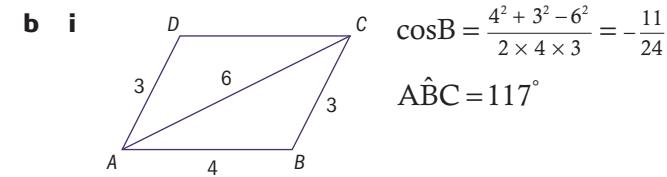
f  $-2\overrightarrow{ED} - \overrightarrow{AF} + \overrightarrow{AB} = \overrightarrow{CF} + \overrightarrow{FA} + \overrightarrow{AB} = \overrightarrow{CB}$   
 (other answers are possible in this question)

3 a i  $\overrightarrow{AC} = \mathbf{u} + \mathbf{v}$

ii  $\overrightarrow{HB} = \mathbf{u} - \mathbf{w} + \mathbf{u} = 2\mathbf{u} - \mathbf{w}$

iii  $\overrightarrow{CE} = -\mathbf{v} - \mathbf{u} + \mathbf{w} - \mathbf{u} - \mathbf{v} = \mathbf{w} - 2\mathbf{u} - 2\mathbf{v}$

iv  $\overrightarrow{AF} = \mathbf{w} - \mathbf{v}$



ii area  $ABCD = 2 \left( \frac{1}{2} \times 4 \times 3 \times \sin 117^\circ \right) = 10.7$  sq. units

- 4 a  $3\mathbf{x} - \mathbf{u} = 6\mathbf{v} + 2\mathbf{u} \Rightarrow 3\mathbf{x} = 6\mathbf{v} + 3\mathbf{u}$   
 $\Rightarrow \mathbf{x} = 2\mathbf{v} + \mathbf{u}$
- b  $2(\mathbf{x} - \mathbf{u}) + 3(\mathbf{u} - \mathbf{v}) = 0 \Rightarrow 2\mathbf{x} - 2\mathbf{u} + 3\mathbf{u} - 3\mathbf{v} = 0$   
 $\mathbf{x} = \frac{1}{2}(3\mathbf{v} - \mathbf{u})$
- c  $\frac{1}{2}(\mathbf{x} - \mathbf{u}) = \frac{1}{3}(\mathbf{x} + \mathbf{v}) \Rightarrow 3\mathbf{x} - 3\mathbf{u} = 2\mathbf{x} + 2\mathbf{v}$   
 $\Rightarrow \mathbf{x} = 3\mathbf{u} + 2\mathbf{v}$

### Exercise 11C

1 A(-1, 3) C(5, 4) I(7, 8)

a i  $\overrightarrow{AB} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2} \left[ \begin{pmatrix} 5 \\ 4 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right] = \frac{1}{2} \begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0.5 \end{pmatrix}$

ii  $\overrightarrow{AE} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{CI}$

$$= \begin{pmatrix} 3 \\ 0.5 \end{pmatrix} + \frac{1}{2} \left[ \begin{pmatrix} 7 \\ 8 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \end{pmatrix} \right] = \begin{pmatrix} 4 \\ 2.5 \end{pmatrix}$$

iii  $\overrightarrow{CD} = \frac{1}{2}\overrightarrow{CI} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

b i  $\overrightarrow{BF} = \overrightarrow{BA} + \overrightarrow{AF} = \begin{pmatrix} -3 \\ -0.5 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -2\mathbf{i} + 1.5\mathbf{j}$

ii  $\overrightarrow{CH} = \overrightarrow{CB} + \overrightarrow{BH} = \begin{pmatrix} -3 \\ -0.5 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} = -\mathbf{i} + 3.5\mathbf{j}$

iii  $\overrightarrow{DG} = \overrightarrow{DF} + \overrightarrow{FG} = \begin{pmatrix} -6 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -5\mathbf{i} + \mathbf{j}$

c i  $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3.5 \end{pmatrix}$

$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$

$\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 2.5 \end{pmatrix} = \begin{pmatrix} 3 \\ 5.5 \end{pmatrix}$

$\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$

$\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{AG} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$

**2** P(0, 2, -1) Q(2, 1, 1)

$$\mathbf{a} \quad \overrightarrow{OP} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \quad \overrightarrow{OQ} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{b} \quad \overrightarrow{PQ} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$$

$$\mathbf{3} \quad \mathbf{a} \quad \overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\mathbf{b} \quad \overrightarrow{AD} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

$$\mathbf{c} \quad \overrightarrow{AE} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$\mathbf{d} \quad \overrightarrow{AG} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CG} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$$

$$\mathbf{e} \quad \overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -3 \end{pmatrix}$$

$$\mathbf{f} \quad \overrightarrow{BH} = \overrightarrow{BD} + \overrightarrow{DH} = \begin{pmatrix} 1 \\ -3 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}$$

**4** P(-3, 1) Q(5, 7) R(-1, 5)

$$\mathbf{a} \quad \overrightarrow{OP} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \overrightarrow{OQ} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} \quad \overrightarrow{OR} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

$$\mathbf{b} \quad M(1, 4) N(-2, 3)$$

$$\mathbf{c} \quad \overrightarrow{QR} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \end{pmatrix}$$

$$\overrightarrow{MN} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix} \therefore \overrightarrow{QR} = 2 \overrightarrow{MN} \text{ (QED)}$$

**Exercise 11D**

$$\mathbf{1} \quad \mathbf{a} \quad \mathbf{u} + (-\mathbf{u}) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} -u_1 \\ -u_2 \end{pmatrix} = \begin{pmatrix} u_1 - u_1 \\ u_2 - u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0} \text{ (QED)}$$

$$\mathbf{b} \quad \mathbf{u} + (\mathbf{v} + \mathbf{w}) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \left[ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix} \\ = \begin{pmatrix} u_1 + (v_1 + w_1) \\ u_2 + (v_2 + w_2) \end{pmatrix} = \begin{pmatrix} (u_1 + v_1) + w_1 \\ (u_2 + v_2) + w_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ = (\mathbf{u} + \mathbf{v}) + \mathbf{w} \text{ (QED)}$$

$$\mathbf{c} \quad \alpha(\beta\mathbf{u}) = \alpha \begin{pmatrix} \beta u_1 \\ \beta u_2 \end{pmatrix} = \begin{pmatrix} \alpha\beta u_1 \\ \alpha\beta u_2 \end{pmatrix} = \alpha\beta \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (\alpha\beta)\mathbf{u}$$

$$\alpha(\beta\mathbf{u}) = \begin{pmatrix} \alpha\beta u_1 \\ \alpha\beta u_2 \end{pmatrix} = \begin{pmatrix} \beta\alpha u_1 \\ \beta\alpha u_2 \end{pmatrix} = \beta \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \end{pmatrix} = \beta(\alpha\mathbf{u})$$

(QED)

$$\mathbf{d} \quad \alpha(\mathbf{u} + \mathbf{v}) = \alpha \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} = \begin{pmatrix} \alpha(u_1 + v_1) \\ \alpha(u_2 + v_2) \end{pmatrix} = \begin{pmatrix} \alpha u_1 + \alpha v_1 \\ \alpha u_2 + \alpha v_2 \end{pmatrix} \\ = \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \end{pmatrix} + \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \end{pmatrix} = \alpha \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \alpha \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \alpha\mathbf{u} + \alpha\mathbf{v} \text{ (QED)}$$

$$\mathbf{e} \quad (\alpha + \beta)\mathbf{u} = (\alpha + \beta) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} (\alpha + \beta)u_1 \\ (\alpha + \beta)u_2 \end{pmatrix} = \begin{pmatrix} \alpha u_1 + \beta u_1 \\ \alpha u_2 + \beta u_2 \end{pmatrix} \\ = \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \end{pmatrix} + \begin{pmatrix} \beta u_1 \\ \beta u_2 \end{pmatrix} = \alpha \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \beta \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \alpha\mathbf{u} + \beta\mathbf{u} \text{ (QED)}$$

$$\mathbf{f} \quad 0\mathbf{u} = 0 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0u_1 \\ 0u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0} \text{ (QED)}$$

$$\mathbf{g} \quad \alpha\mathbf{0} = \alpha \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha 0 \\ \alpha 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0} \text{ (QED)}$$

$$\mathbf{2} \quad \mathbf{a} \quad 2 \begin{pmatrix} x \\ y \end{pmatrix} - 3 \begin{pmatrix} y \\ x \end{pmatrix} = 5 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$2x - 3y = 5$$

$$2y - 3x = -10, \quad x = 4, \quad y = 1$$

$$\mathbf{b} \quad 2 \left( \begin{pmatrix} 2 \\ y \end{pmatrix} - \begin{pmatrix} x \\ 2 \end{pmatrix} \right) - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 0 \Rightarrow 2 \begin{pmatrix} 2-x \\ y-2 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 0$$

$$4 - 2x - 1 = 0$$

$$2y - 4 - 3 = 0 \quad \therefore x = \frac{3}{2}, \quad y = \frac{7}{2}$$

$$\mathbf{3} \quad \mathbf{a} \quad \mathbf{u} + (\mathbf{v} + 2\mathbf{u}) = 3\mathbf{u} + \mathbf{v}$$

$$\mathbf{b} \quad (\mathbf{u} - \mathbf{v}) + 2(\mathbf{v} - 2\mathbf{u}) = \mathbf{u} - \mathbf{v} + 2\mathbf{v} - 4\mathbf{u} = -3\mathbf{u} + \mathbf{v}$$

$$\mathbf{c} \quad 3 \left( \frac{1}{6}(\mathbf{u} - \mathbf{v}) + \frac{1}{3}(\mathbf{v} - \mathbf{u}) \right) = \frac{1}{2}(\mathbf{u} - \mathbf{v}) + (\mathbf{v} - \mathbf{u}) = -\frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v}$$

$$\mathbf{4} \quad \mathbf{a} = 2\mathbf{i} - 3\mathbf{j} \quad \mathbf{b} = -\mathbf{i} - 2\mathbf{j}$$

$$\alpha\mathbf{a} + \beta\mathbf{b} = 3\mathbf{i} - \mathbf{j}$$

$$\therefore \alpha(2\mathbf{i} - 3\mathbf{j}) + \beta(-\mathbf{i} - 2\mathbf{j}) = 3\mathbf{i} - \mathbf{j}$$

$$2\alpha - \beta = 3, \quad -3\alpha - 2\beta = -1 \quad \therefore \alpha = 1, \quad \beta = -1$$

$$6\mathbf{i} - 2\mathbf{j} = 2\mathbf{a} - 2\mathbf{b}$$

**Exercise 11E**

$$\mathbf{1} \quad \mathbf{a} \quad \mathbf{v} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \quad |\mathbf{v}| = \sqrt{26} \quad \frac{1}{\sqrt{26}} \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{26}} \\ \frac{5}{\sqrt{26}} \end{pmatrix}$$

$$\mathbf{b} \quad \mathbf{v} = \begin{pmatrix} 5 \\ 12 \end{pmatrix} \quad |\mathbf{v}| = 13 \quad \frac{1}{13} \begin{pmatrix} 5 \\ 12 \end{pmatrix} = \begin{pmatrix} \frac{5}{13} \\ \frac{12}{13} \end{pmatrix}$$

$$\mathbf{c} \quad \mathbf{v} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \quad |\mathbf{v}| = 3 \quad \frac{1}{3} \begin{pmatrix} -3 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\mathbf{d} \quad \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad |\mathbf{v}| = \sqrt{2} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

**2 a**  $\mathbf{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$   $|\mathbf{v}| = \sqrt{5}$   $\pm \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$  or  $\begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}$

**b**  $\mathbf{v} = \begin{pmatrix} -5 \\ -2 \end{pmatrix}$   $|\mathbf{v}| = \sqrt{29}$   $\pm \frac{1}{\sqrt{29}} \begin{pmatrix} -5 \\ -2 \end{pmatrix} = \begin{pmatrix} -\frac{5}{\sqrt{29}} \\ -\frac{2}{\sqrt{29}} \end{pmatrix}$  or  $\begin{pmatrix} \frac{5}{\sqrt{29}} \\ \frac{2}{\sqrt{29}} \end{pmatrix}$

**c**  $\mathbf{v} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$   $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$  or  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

**d**  $\mathbf{v} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$   $|\mathbf{v}| = \sqrt{2}$   $\pm \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$  or  $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

**3**  $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$   $|\mathbf{u}| = \sqrt{13}$   $\mathbf{v} = \frac{1}{\sqrt{13}} \mathbf{u} = \begin{pmatrix} \frac{2}{\sqrt{13}} \\ \frac{-3}{\sqrt{13}} \end{pmatrix}$

**4 a**  $|\mathbf{u}| = \sqrt{13}$   $\mathbf{v} = \pm \frac{2}{\sqrt{13}}$   $\mathbf{u} = \begin{pmatrix} \frac{4}{\sqrt{13}} \\ \frac{6}{\sqrt{13}} \end{pmatrix}$  or  $\begin{pmatrix} -\frac{4}{\sqrt{13}} \\ -\frac{6}{\sqrt{13}} \end{pmatrix}$

**b**  $|\mathbf{u}| = 3$   $\mathbf{v} = \pm \frac{2}{3}$   $\mathbf{u} = \begin{pmatrix} -\frac{4}{3} \\ \frac{2\sqrt{5}}{3} \end{pmatrix}$  or  $\begin{pmatrix} \frac{4}{3} \\ -\frac{2\sqrt{5}}{3} \end{pmatrix}$

**c**  $|\mathbf{u}| = \sqrt{26}$   $\mathbf{v} = \pm \frac{13}{\sqrt{26}}$   $\mathbf{u} = \pm \sqrt{\frac{13}{2}} \mathbf{u} = \begin{pmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix}$  or  $\begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{-3}{\sqrt{13}} \end{pmatrix}$

**5**  $\mathbf{u} = -4\mathbf{i} - 6\mathbf{j}$   $|\mathbf{u}| = \sqrt{52} = 2\sqrt{13}$

$\mathbf{w} = \frac{1}{2} \mathbf{u} = -2\mathbf{i} - 3\mathbf{j}$

**6**  $\mathbf{u} = \mathbf{i} - 3\mathbf{j}$   $|\mathbf{u}| = \sqrt{10}$   $\mathbf{t} = \pm \frac{5}{\sqrt{10}} \mathbf{u}$

$\mathbf{t} = \frac{5}{\sqrt{10}} \mathbf{i} - \frac{15}{\sqrt{10}} \mathbf{j}$  or  $\mathbf{t} = -\frac{5}{\sqrt{10}} \mathbf{i} + \frac{15}{\sqrt{10}} \mathbf{j}$

**7**  $\mathbf{u} = \frac{1}{\sqrt{v_1^2 + v_2^2}} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\sqrt{v_1^2 + v_2^2}} \mathbf{v}$   $\therefore \mathbf{u} = k\mathbf{v}$  ( $k > 0$ )

$\therefore \mathbf{u}$  is in the same direction as  $\mathbf{v}$  (QED)

$|\mathbf{u}|^2 = \frac{v_1^2}{v_1^2 + v_2^2} + \frac{v_2^2}{v_1^2 + v_2^2} = \frac{v_1^2 + v_2^2}{v_1^2 + v_2^2} = 1$   $\therefore |\mathbf{u}| = 1$

$\therefore \mathbf{u}$  has magnitude 1 (QED)

**8**  $\mathbf{u} = \pm \frac{\mathbf{m}}{|\mathbf{v}|} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \pm \begin{pmatrix} \frac{mv_1}{|\mathbf{v}|} \\ \frac{mv_2}{|\mathbf{v}|} \end{pmatrix}$

$|\mathbf{u}|^2 = \frac{m^2 v_1^2}{|\mathbf{v}|^2} + \frac{m^2 v_2^2}{|\mathbf{v}|^2} = \frac{m^2 (v_1^2 + v_2^2)}{v_1^2 + v_2^2} = m^2$

$\therefore |\mathbf{u}| = m$  (QED)

### Exercise 11F

**1 a** A(2, 5) B(5, 6) C(4, 2) D(1, 1)

**b**  $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$\overrightarrow{AC} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\overrightarrow{AD} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$$

**c**  $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = -\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -4 \end{pmatrix} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$

**2** A(2, 6) B(-2, 4)

**a**  $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$

**b**  $AB = \sqrt{(-2-2)^2 + (4-6)^2} = \sqrt{20} = 2\sqrt{5}$   
 $\therefore |\overrightarrow{AB}| = 2\sqrt{5}$

**c** M(0, 5) M is the midpoint of AB

**d** Let P( $x_1, y_1$ )  $\overrightarrow{AP} = \begin{pmatrix} x_1 - 2 \\ y_1 - 6 \end{pmatrix}$   $\overrightarrow{PB} = \begin{pmatrix} -2 - x_1 \\ 4 - y_1 \end{pmatrix}$

$$\overrightarrow{AP} = 2\overrightarrow{PB} \therefore x_1 - 2 = 2(-2 - x_1), y_1 - 6 = 2(4 - y_1)$$

$$x_1 - 2 = -4 - 2x_1, y_1 - 6 = 8 - 2y_1$$

$$3x_1 = -2 \quad 3y_1 = 14$$

$$x_1 = \frac{-2}{3} \quad y_1 = \frac{14}{3}$$

$$\therefore P\left(\frac{-2}{3}, \frac{14}{3}\right)$$

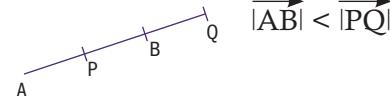
Let Q( $x_2, y_2$ ),  $\overrightarrow{AQ} = \begin{pmatrix} x_2 - 2 \\ y_2 - 6 \end{pmatrix}$   $\overrightarrow{QB} = \begin{pmatrix} -2 - x_2 \\ 4 - y_2 \end{pmatrix}$   
 $\overrightarrow{AQ} = -2\overrightarrow{QB}$

$$\therefore x_2 - 2 = -2(-2 - x_2), y_2 - 6 = -2(4 - y_2)$$

$$x_2 - 2 = 4 + 2x_2, y_2 - 6 = -8 + 2y_2$$

$$x_2 = -6 \quad y_2 = 2$$

$$\therefore Q(-6, 2)$$



$\overrightarrow{PQ}$  has greater magnitude than  $\overrightarrow{AB}$

**3** P(4, -1) Q(6, -3) R(2, 1)

$$\overrightarrow{PQ} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \quad \overrightarrow{PR} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \quad \overrightarrow{PR} = -1\overrightarrow{PQ}$$

$\therefore \overrightarrow{PQ}$  and  $\overrightarrow{PR}$  are collinear

$\therefore P, Q, R$  are collinear (QED)

**4** A( $a, a-1$ ) B(2, 2 $a$ ) C(0, 3 $a$ )

$$\overrightarrow{AB} = \begin{pmatrix} 2-a \\ a+1 \end{pmatrix} \quad \overrightarrow{AC} = \begin{pmatrix} -a \\ 2a+1 \end{pmatrix} \quad \overrightarrow{AC} = k \overrightarrow{AB}$$

$$\therefore -a = k(2-a)$$

$$2a+1 = k(a+1) \quad \therefore \frac{-a}{2a+1} = \frac{2-a}{a+1}$$

$$-a^2 - a = 4a + 2 - 2a^2 - a$$

$$a^2 - 4a - 2 = 0$$

$$a = \frac{4 \pm \sqrt{16+8}}{2}, a = 2 \pm \sqrt{6}$$

5 S(2, -3) U(-1, 2) N(1, -4)

$$\overrightarrow{SU} = \begin{pmatrix} -3 \\ 5 \end{pmatrix} \quad \overrightarrow{SN} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \overrightarrow{SN} \neq k\overrightarrow{SU}$$

$\therefore$  S, U, N are not collinear

$\therefore$  they form a triangle (QED)

6 P(a, b) Q(c, d) R(e, f)

$$\overrightarrow{PQ} = \begin{pmatrix} c-a \\ d-b \end{pmatrix} \quad \overrightarrow{PR} = \begin{pmatrix} e-a \\ f-b \end{pmatrix}$$

$$\frac{f-b}{d-b} = \frac{e-a}{c-a} \Rightarrow e-a = \frac{(f-b)(c-a)}{d-b}$$

$$\therefore \overrightarrow{PR} = \begin{pmatrix} \frac{(f-b)(c-a)}{d-b} \\ \frac{(f-b)(d-b)}{d-b} \end{pmatrix} = \frac{f-a}{d-b} \begin{pmatrix} c-a \\ d-b \end{pmatrix}$$

$$\therefore \overrightarrow{PR} = \frac{f-a}{d-b} \overrightarrow{PQ} \quad \therefore \overrightarrow{PR} = k \overrightarrow{PQ}$$

$\therefore$  P, Q, R are collinear points

7 a  $\overrightarrow{AB} = \begin{pmatrix} \sin 2x - \sin x \\ \cos 2x - (-1 + \cos x) \end{pmatrix}$

$$= \begin{pmatrix} 2 \sin x \cos x - \sin x \\ 2 \cos^2 x - 1 + 1 - \cos x \end{pmatrix}$$

$$= \begin{pmatrix} \sin x (2 \cos x - 1) \\ 2 \cos^2 x - \cos x \end{pmatrix}$$

$$= \begin{pmatrix} \sin x (2 \cos x - 1) \\ \cos x (2 \cos x - 1) \end{pmatrix}$$

$$= 2 \cos x - 1 \begin{pmatrix} \sin x \\ \cos x \end{pmatrix}$$

Therefore for any value of  $x$ ,  $\overrightarrow{AB}$  is collinear

with  $\begin{pmatrix} \sin x \\ \cos x \end{pmatrix}$

b  $|\overrightarrow{AB}| = |2 \cos x - 1| \sqrt{\sin^2 x + \cos^2 x}$

$$= |2 \cos x - 1| \sqrt{1}$$

$$= |2 \cos x - 1|$$

### Exercise 11G

1 a  $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$  b  $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

c  $\mathbf{u} + \mathbf{v} = -3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$

d  $-3\mathbf{u} = 6\mathbf{i} - 9\mathbf{j} - 3\mathbf{k}$

e  $4\mathbf{u} - 2\mathbf{v} = -8\mathbf{i} + 12\mathbf{j} + 4\mathbf{k} - (-2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k})$   
 $= -6\mathbf{i} + 8\mathbf{j} + 10\mathbf{k}$

f  $-2(\mathbf{u} - \mathbf{v}) = -2(-\mathbf{i} + \mathbf{j} + 4\mathbf{k}) = 2\mathbf{i} - 2\mathbf{j} - 8\mathbf{k}$

2 a  $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$  b  $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$  c  $\mathbf{c} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$

a  $\mathbf{a} + \mathbf{b} + \mathbf{c} = \begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix}$  b  $2\mathbf{a} - \mathbf{b} + \mathbf{c} = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix}$

c  $2(a-b) - 3c = 2 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ -7 \end{pmatrix}$

d  $\frac{1}{2}(\mathbf{a} - 3\mathbf{b}) = \frac{1}{2} \begin{pmatrix} 5 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ -\frac{3}{2} \\ \frac{5}{2} \end{pmatrix}$

e  $|\mathbf{a}| = \sqrt{14}$  f  $|\mathbf{b}| = 3$

g  $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} \therefore |\mathbf{a} + \mathbf{b}| = \sqrt{35}$

h  $\mathbf{a} - \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \therefore |\mathbf{a} - \mathbf{b}| = \sqrt{11}$

3 A(0, 2, 1) B(-1, -1, -2) C(1, -3, 0)

a  $\overrightarrow{AB} = \begin{pmatrix} -1 \\ -3 \\ -3 \end{pmatrix} \quad \overrightarrow{AC} = \begin{pmatrix} 1 \\ -5 \\ -1 \end{pmatrix}$

b  $\overrightarrow{AB} - \overrightarrow{AC} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix} \quad \overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

4  $\nu = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

a  $\mathbf{u} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$  b  $\pm \begin{pmatrix} 0 \\ \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$  c  $\begin{pmatrix} 0 \\ \sqrt{5} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$

5 A(4, -1, 3) C(0, -2, 5) D(5, 1, 6) G(1, -4, 6)

a  $\overrightarrow{AC} = \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} \quad \overrightarrow{AD} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \overrightarrow{CG} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

b  $\overrightarrow{AB} = \overrightarrow{AC} - \overrightarrow{AD} = \begin{pmatrix} -5 \\ -3 \\ -1 \end{pmatrix} \quad \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix}$

$\overrightarrow{AE} = \overrightarrow{CG} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE} = \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix}$

$\overrightarrow{BF} = \overrightarrow{CG} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \overrightarrow{OF} = \overrightarrow{OB} + \overrightarrow{BF} = \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$

$\overrightarrow{AH} = \overrightarrow{AD} - \overrightarrow{CG} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \quad \overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{AH} = \begin{pmatrix} 6 \\ -1 \\ 7 \end{pmatrix}$

**Exercise 11H**

1  $\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$   $r = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$x = 1 + \lambda, y = 3 + 2\lambda$$

$$x - 1 = \frac{y - 3}{2}$$

$$2x - 2 = y - 3$$

$$y = 2x + 1$$

2  $r = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$$x = 1 + 2\lambda, y = -1 - \lambda, z = 1 + 3\lambda$$

$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-1}{3}$$

3  $\frac{x+1}{3} = \frac{2y}{3} = z-1 \Rightarrow \frac{x+1}{3} = \frac{y}{\frac{3}{2}} = \frac{z-1}{1}$

$$(-1, 0, 1) \quad \begin{pmatrix} 3 \\ \frac{3}{2} \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$$

4 a eg use  $\lambda = 0, 1, -1$

$$(1, 1, -1) \quad (0, 1, 2) \quad (2, 1, -4)$$

(other solutions are possible)

b  $x = 1 - \lambda \quad y = 1 \quad z = -1 + 3\lambda$

At P,  $y = 3 \neq 1 \therefore P$  does not lie on L (QED)

c  $r = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$

5 a eg use  $k = 0, 1 \quad (1, 0, 2) \quad (2, -1, 2)$

b direction of line =  $\mathbf{i} - \mathbf{j}$

$$\mathbf{u} = \pm \frac{4}{\sqrt{2}}(\mathbf{i} - \mathbf{j}) = \pm (2\sqrt{2}\mathbf{i} - 2\sqrt{2}\mathbf{j}) = \pm \begin{pmatrix} \frac{2}{\sqrt{2}} \\ -\frac{2}{\sqrt{2}} \\ 0 \end{pmatrix}$$

**Exercise 11I**

1  $\mathbf{u} \cdot \mathbf{v} = 1.5 \times 4 \times \cos 30^\circ = 3\sqrt{3}$

2  $\mathbf{u} \cdot (-\mathbf{v}) = |\mathbf{u}| |\mathbf{-v}| \cos (\pi - \theta)$

$$= |\mathbf{u}| |\mathbf{v}| \cos (\pi - \theta)$$

$$= -|\mathbf{u}| |\mathbf{v}| \cos \theta$$

$$= -(\mathbf{u} \cdot \mathbf{v})$$

$$(-\mathbf{u}) \cdot \mathbf{v} = -|\mathbf{u}| |\mathbf{v}| \cos (\pi - \theta)$$

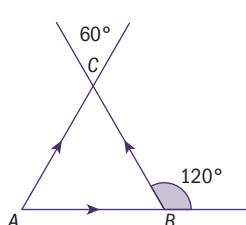
$$= |\mathbf{u}| |\mathbf{v}| \cos (\pi - \theta)$$

$$= -|\mathbf{u}| |\mathbf{v}| \cos \theta$$

$$= -(\mathbf{u} \cdot \mathbf{v})$$

$$\therefore \mathbf{u} \cdot (-\mathbf{v}) = -(\mathbf{u} \cdot \mathbf{v}) = (-\mathbf{u}) \cdot \mathbf{v} \quad (\text{QED})$$

3



Let the length of the sides be  $x$

$$\overrightarrow{AB} \cdot \overrightarrow{BC} = x^2 \cos 120^\circ = -\frac{1}{2}x^2$$

$$\overrightarrow{BC} \cdot \overrightarrow{AC} = x^2 \cos 60^\circ = \frac{1}{2}x^2$$

$$\therefore \overrightarrow{AB} \cdot \overrightarrow{BC} + \overrightarrow{BC} \cdot \overrightarrow{AC} = -\frac{1}{2}x^2 + \frac{1}{2}x^2 = 0$$

4 a  $\overrightarrow{AB} \cdot \overrightarrow{AC} = xy \cos \alpha = x^2$

b  $\overrightarrow{CA} \cdot \overrightarrow{CB}$

$$= y\sqrt{y^2 - x^2} \cos\left(\frac{\pi}{2} - \alpha\right) = y\sqrt{y^2 - x^2} \sin \alpha = y^2 \sin^2 \alpha$$

c  $\overrightarrow{AC} \cdot \overrightarrow{CB}$

$$= y\sqrt{y^2 - x^2} \cos\left(\frac{\pi}{2} + \alpha\right) = -y\sqrt{y^2 - x^2} \sin \alpha = -y^2 \sin^2 \alpha$$

5 Area = 4

$$\therefore \frac{1}{2} \times 2 \times 5 \sin \theta = 4$$

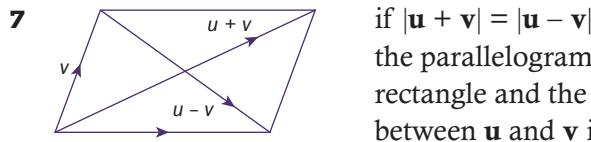
$$\therefore \sin \theta = \frac{4}{5}$$

$$\therefore \cos \theta = \pm \frac{3}{5}$$

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = 2 \times 5 \cos \theta = \pm 6$$

6 Angle between  $\mathbf{u}$  and  $\mathbf{u}$  = 0

$$\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}| |\mathbf{u}| \cos 0 = |\mathbf{u}|^2 \quad (\text{QED})$$



if  $|\mathbf{u} + \mathbf{v}| = |\mathbf{u} - \mathbf{v}|$

the parallelogram is a rectangle and the angle between  $\mathbf{u}$  and  $\mathbf{v}$  is  $\frac{\pi}{2}$

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \frac{\pi}{2} = 0 \quad (\text{QED})$$

8 diagonal AG =  $\sqrt{7^2 + 2^2 + 3^2} = \sqrt{62}$

a  $OB = OC = \frac{\sqrt{62}}{2} \cos BOC = \frac{\left(\frac{\sqrt{62}}{2}\right)^2 + \left(\frac{\sqrt{62}}{2}\right)^2 - 2^2}{2 \times \frac{\sqrt{62}}{2} \times \frac{\sqrt{62}}{2}} = \frac{27}{31}$

$$\overrightarrow{OB} \cdot \overrightarrow{OC} = \left(\frac{\sqrt{62}}{2}\right) \left(\frac{\sqrt{62}}{2}\right) \left(\frac{27}{31}\right) = \frac{27}{2}$$

b  $OA = OE = \frac{\sqrt{62}}{2} \cos AOE = \frac{\left(\frac{\sqrt{62}}{2}\right)^2 + \left(\frac{\sqrt{62}}{2}\right)^2 - 3^2}{2 \times \frac{\sqrt{62}}{2} \times \frac{\sqrt{62}}{2}} = \frac{22}{31}$

$$\overrightarrow{OA} \cdot \overrightarrow{OE} = \left(\frac{\sqrt{62}}{2}\right) \left(\frac{\sqrt{62}}{2}\right) \left(\frac{22}{31}\right) = 11$$

**Exercise 11J**

1 a  $\mathbf{u} \cdot \mathbf{v} = -12 + 24 = 12$

b  $\mathbf{u} \cdot \mathbf{v} = -1 - 3 + 10 = 6$

2 A(-1, 3, 2) B(-1, 1, 2) C(1, -1, 1)

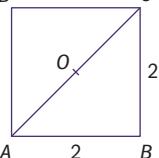
$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix} \quad \overrightarrow{BC} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} \quad \overrightarrow{AC} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$$

$$\overrightarrow{AB} \cdot \overrightarrow{BC} = 0 + 4 - 4 = 0$$

$$\overrightarrow{AC} \cdot \overrightarrow{BC} = 4 + 8 - 3 = 9$$

- 3 a**  $0(0, 0, 0)$   $A(0, 0, 1)$   $B(1, 0, 1)$   $C(1, 0, 0)$   
 $D(0, 1, 0)$   $E(0, 1, 1)$   $F(1, 1, 1)$   $G(1, 1, 0)$

**b**  $\overrightarrow{OF} \cdot \overrightarrow{OG} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 2$   
 $\overrightarrow{AF} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$   $\overrightarrow{BG} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$   $\overrightarrow{AF} \cdot \overrightarrow{BG} = 1$

- 4 a**   
 $AC = 2\sqrt{2}$   
 $\therefore OA = OB = OC = OD = \sqrt{2}$

$\text{Volume} = \frac{1}{3} \times 4 \times \text{OE} = \frac{8}{3} \therefore \text{OE} = 2$

$A(0, \sqrt{2}, 0) \quad B(\sqrt{2}, 0, 0) \quad C(0, \sqrt{2}, 0)$   
 $D(-\sqrt{2}, 0, 0) \quad E(0, 0, 2)$

**b**  $\overrightarrow{EA} = \begin{pmatrix} 0 \\ -\sqrt{2} \\ -2 \end{pmatrix}$   $|\overrightarrow{EA}| = \sqrt{6}$

$\overrightarrow{EB} = \begin{pmatrix} \sqrt{2} \\ 0 \\ -2 \end{pmatrix}$   $\overrightarrow{EA} \cdot \overrightarrow{EB} = 4$

**c**  $\cos A\hat{E}B = \frac{6+6-4}{2\sqrt{6}\sqrt{6}} = \frac{8}{12}$   
 $A\hat{E}B = 48.2^\circ$

### Exercise 11K

**1**  $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$   $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$

$\mathbf{u} \cdot \mathbf{v} = 2 - 6 = -4 \quad |\mathbf{u}| = \sqrt{13} \quad |\mathbf{v}| = \sqrt{5}$

$\cos \theta = \frac{-4}{\sqrt{13}\sqrt{5}} \quad \therefore \theta = 120^\circ$

**2**  $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$   $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$

$\mathbf{u} \cdot \mathbf{v} = 2 + 2 + 1 = 5 \quad |\mathbf{u}| = \sqrt{6} \quad |\mathbf{v}| = \sqrt{6}$

$\cos \theta = \frac{5}{6} \quad \therefore \theta = 33.6^\circ$

**3**  $A(-1, 1, 1)$   $B(1, -1, 2)$   $C(2, 3, -1)$

**a**  $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$   $\overrightarrow{AC} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$

$\overrightarrow{AB} \cdot \overrightarrow{AC} = 6 - 4 - 2 = 0$

$\therefore \cos \theta = 0 \quad \therefore \theta = 90^\circ$

**b**  $\overrightarrow{BC} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$   $\overrightarrow{BC} \cdot \overrightarrow{AC} = 3 + 8 + 6 = 17$

$|\overrightarrow{BC}| = \sqrt{26} \quad |\overrightarrow{AC}| = \sqrt{17}$

$\cos \theta = \frac{17}{\sqrt{26}\sqrt{17}} \quad \therefore \theta = 36.0^\circ$

**4**  $\mathbf{u} \cdot \mathbf{v} > 0 \therefore a(a-2) + 3(a-4) > 0$

$a^2 + a - 12 > 0$

$\text{When } (a-3)(a+4) > 0$

$a = 3 \text{ or } -4 \quad \therefore a < -4 \text{ or } a > 3$

**5**  $\mathbf{u} = \sin 3\alpha \mathbf{i} - \cos 3\alpha \mathbf{j} + 2\mathbf{k}$

$\mathbf{v} = \cos \alpha \mathbf{i} - \sin \alpha \mathbf{j} - 2\mathbf{k}$

**a**  $\mathbf{u} \cdot \mathbf{v} = \sin 3\alpha \cos \mu + \cos 3\alpha \sin \alpha - 4$   
 $= \sin 4\alpha - 4$

$|\mathbf{u}|^2 = \sin^2 3\alpha + \cos^2 3\alpha + 4 = 5 \quad \therefore |\mathbf{u}| = \sqrt{5}$

$|\mathbf{v}|^2 = \cos^2 \alpha + \sin^2 \alpha + 4 = 5 \quad \therefore |\mathbf{v}| = \sqrt{5}$

$\therefore \cos \theta = \frac{\sin 4\alpha - 4}{5}$

**b**  $\cos 150^\circ = \frac{\sin 4\alpha - 4}{5}$

$-\frac{\sqrt{3}}{2} = \frac{\sin 4\alpha - 4}{5}$

$\sin 4\alpha = -0.3301$

$4\alpha = 3.478, 5.947, 9.761, 12.230, 16.044, 18.513, 22.328, 27.796$

$\alpha = 0.870, 1.49, 2.44, 3.06, 4.01, 4.63, 5.58, 6.20$

**c**  $\sin 4\alpha < 4 \quad \therefore \sin 4\alpha - 4 < 0$

$\therefore \cos \theta < 0 \quad \therefore \theta \text{ is obtuse} \quad (\text{QED})$

**6** Let  $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$   $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$   $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$   $\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

$\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} \quad \mathbf{c} + \mathbf{d} = \begin{pmatrix} c_1 + d_1 \\ c_2 + d_2 \\ c_3 + d_3 \end{pmatrix}$

$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) = (a_1 + b_1)(c_1 + d_1) + (a_2 + b_2)(c_2 + d_2) + (a_3 + b_3)(c_3 + d_3)$

$= a_1 c_1 + a_1 d_1 + b_1 c_1 + b_1 d_1 + a_2 c_2 + a_2 d_2 + b_2 c_2 + b_2 d_2 + a_3 c_3 + a_3 d_3 + b_3 c_3 + b_3 d_3$

$= (a_1 c_1 + a_2 c_2 + a_3 c_3) + (a_1 d_1 + a_2 d_2 + a_3 d_3) + (b_1 c_1 + b_2 c_2 + b_3 c_3) + (b_1 d_1 + b_2 d_2 + b_3 d_3)$

$= \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d} \quad (\text{QED})$

### Exercise 11L

**1**  $\mathbf{u} = \mathbf{i} - \mathbf{j} - \mathbf{k}$   $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$   $\mathbf{w} = \mathbf{i} - \mathbf{k}$

**a**  $\mathbf{u} \times \mathbf{v} = -3\mathbf{i} - 2\mathbf{j} - \mathbf{k} = \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix}$

**b**  $\mathbf{v} \times \mathbf{w} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

**c**  $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = -4\mathbf{j} + 4\mathbf{k} = \begin{pmatrix} 0 \\ -4 \\ 4 \end{pmatrix}$

**d**  $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}$

**e**  $\mathbf{u} \times \mathbf{w} = \mathbf{i} + \mathbf{k} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

**f**  $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{u} \times \mathbf{w}) = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$

**g**  $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{w}) = (2\mathbf{i} - 3\mathbf{j}) \times (-\mathbf{j}) = -2\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$

**2**  $\mathbf{u} = \mathbf{i} - \mathbf{j} - \mathbf{k}$     $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$     $\mathbf{w} = \mathbf{i} - \mathbf{k}$

From qn. 1,  $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = -4\mathbf{j} + 4\mathbf{k}$

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$$

$\therefore \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$

$\therefore$  not associative (QED)

**3**  $\mathbf{w} \cdot \mathbf{u} = 0$  and  $\mathbf{w} \cdot \mathbf{v} = 0$

$\therefore w_1 u_1 + w_2 u_2 + w_3 u_3 = 0$  and

$$w_1 v_1 + w_2 v_2 + w_3 v_3 = 0$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} + (u_3 v_1 - u_1 v_3) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

$$\begin{aligned} \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) &= [w_2 (u_1 v_2 - u_2 v_1) - w_3 (u_3 v_1 - u_1 v_3)] \mathbf{i} \\ &\quad + [w_3 (u_2 v_3 - u_3 v_2) - w_1 (u_1 v_2 - u_2 v_1)] \mathbf{j} \\ &\quad + [w_1 (u_3 v_1 - u_1 v_3) - w_2 (u_2 v_3 - u_3 v_2)] \mathbf{k} \\ &= [u_1 (w_2 v_2 + w_3 v_3) - v_1 (w_2 u_2 + w_3 u_3)] \mathbf{i} \\ &\quad + [u_2 (w_3 v_3 + w_1 v_1) - v_2 (w_3 u_3 + w_1 u_1)] \mathbf{j} \\ &\quad + [u_3 (w_1 v_1 + w_2 v_2) - v_3 (w_1 u_1 + w_2 u_2)] \mathbf{k} \\ &= [u_1 (-w_1 v_1) - v_1 (-w_1 u_1)] \mathbf{i} \\ &\quad + [u_2 (-w_2 v_2) - v_2 (-w_2 u_2)] \mathbf{j} \\ &\quad + [u_3 (-w_3 v_3) - v_3 (-w_3 u_3)] \mathbf{k} = 0 \end{aligned}$$

$$\mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = 0 \therefore \mathbf{w} \text{ and } \mathbf{u} \times \mathbf{v} \text{ are collinear}$$

**4** A(-1, 3, 4)   B(5, 7, 5)   C(3, 9, 6)

**a**  $\overrightarrow{AB} = \overrightarrow{AC} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$     $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \begin{pmatrix} -3 \\ 5 \\ 5 \end{pmatrix}$

$$\therefore D(-3, 5, 5)$$

**b**  $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix}$     $\overrightarrow{BC} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{pmatrix} 2 \\ -8 \\ 20 \end{pmatrix}$$

**c** area =  $|\overrightarrow{AB} \times \overrightarrow{BC}| = \sqrt{4 + 64 + 400} = \sqrt{468} = 2\sqrt{117} = 6\sqrt{13}$

**5** **a** A(1, 0, 2) B(2, 3, 3) C(-3, -1, 2) E(2, 1, 4)  
 $\overrightarrow{AD} = \overrightarrow{BC} = \begin{pmatrix} -5 \\ -4 \\ -1 \end{pmatrix}$     $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \begin{pmatrix} -4 \\ -4 \\ 1 \end{pmatrix}$

$$D(-4, -4, 1)$$

**b** Volume =  $(\overrightarrow{AB} \times \overrightarrow{AD}) \cdot \overrightarrow{AE}$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} -5 \\ -4 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 11 \end{pmatrix}$$

$$\begin{aligned} \text{Volume} &= \begin{pmatrix} 1 \\ -4 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1 - 4 + 22 \\ &= 19 \text{ cu. units} \end{aligned}$$

**6** **a**  $\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} + (u_3 v_1 - u_1 v_3) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$   
 $= -(v_2 u_3 - v_3 u_2) \mathbf{i} - (v_3 u_1 - v_1 u_3) \mathbf{j} - (v_1 u_2 - v_2 u_1) \mathbf{k}$

$$\mathbf{v} \times \mathbf{u} = (v_2 u_3 - v_3 u_2) \mathbf{i} + (v_3 u_1 - v_1 u_3) \mathbf{j} + (v_1 u_2 - v_2 u_1) \mathbf{k}$$

$$\therefore \mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u} \quad (\text{QED})$$

**b**  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix}$

$$\begin{aligned} &= [u_2 (v_3 + w_3) - u_3 (v_2 + w_2)] \mathbf{i} + [u_3 (v_1 + w_1) - u_1 (v_2 + w_2)] \mathbf{j} \\ &\quad - [u_1 (v_3 + w_3) - u_2 (v_1 + w_1)] \mathbf{k} \\ &= [(u_2 v_3 - u_3 v_2) + (u_2 w_3 - u_3 w_2)] \mathbf{i} \\ &\quad + [(u_3 v_1 - u_1 v_3) + (u_3 w_1 - u_1 w_3)] \mathbf{j} \\ &\quad + [(u_1 v_2 - u_2 v_1) + (u_1 w_2 - u_2 w_1)] \mathbf{k} \end{aligned}$$

$$\begin{aligned} &= (u_2 v_3 - u_3 v_2) \mathbf{i} + (u_3 v_1 - u_1 v_3) \mathbf{j} \\ &\quad + (u_1 v_2 - u_2 v_1) \mathbf{k} \\ &+ (u_2 w_3 - u_3 w_2) \mathbf{i} + (u_3 w_1 - u_1 w_3) \mathbf{j} \\ &\quad + (u_1 w_2 - u_2 w_1) \mathbf{k} \end{aligned}$$

$$\therefore \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} \quad (\text{QED})$$

**c**  $\mathbf{u} \times \mathbf{v} \cdot \mathbf{u} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

$$\begin{aligned} &= u_1 u_2 v_3 - u_1 u_3 v_2 + u_2 u_3 v_1 - u_1 u_2 v_3 + u_1 u_3 \\ &\quad v_2 - u_2 u_3 v_1 = 0 \end{aligned}$$

$$\therefore \mathbf{u} \times \mathbf{v} \cdot \mathbf{u} = 0 \quad (\text{QED})$$

**d**  $(\lambda \mathbf{u}) \times \mathbf{v} = \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \\ \lambda u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \lambda u_2 v_3 - \lambda u_3 v_2 \\ \lambda u_3 v_1 - \lambda u_1 v_3 \\ \lambda u_1 v_2 - \lambda u_2 v_1 \end{pmatrix}$

$$= \lambda \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} = \lambda (\mathbf{u} \times \mathbf{v})$$

$$\therefore (\lambda \mathbf{u}) \times \mathbf{v} = \lambda (\mathbf{u} \times \mathbf{v}) \quad (\text{QED})$$

**Exercise 11M**

**1** A(-3, 1, 1) B(1, 2, 0) C(1, 1, -2)

**a**  $\vec{AB} = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$   $\vec{AC} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$

$$r = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$$

**b**  $x = -3 + 4\alpha + 4\beta \quad (1)$

$$y = 1 + \alpha \quad (2)$$

$$z = 1 - \alpha - 3\beta \quad (3)$$

**c** From (2)  $\alpha = y - 1$

From (1)  $4\beta = x + 3 - 4(y - 1)$

$$\beta = \frac{x - 4y + 7}{4}$$

sub in (3):  $z = 1 - y + 1 - \frac{3}{4}(x - 4y + 7)$

$$4z = 8 - 4y - 3x + 12y - 21$$

$$3x - 8y + 4z = -13 \text{ (other forms are possible)}$$

**2** P(1, 0, 1)  $\mathbf{n} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$$

$$-x + 3y - z = -2$$

$$x - 3y + z = 2$$

eg (0, 0, 2) (0, - $\frac{2}{3}$ , 0) (2, 0, 0)

**3 a** (-3, 4, 0)  $\mathbf{n} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

$$x - 2y - z = -11$$

**b** P(-1, 1, 1)  $x - 1 = \frac{y-1}{2} = z$

line passes through A(1, 1, 0) and has direction  $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$$\vec{AP} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix}$$

$$-2x + 3y - 4z = 1$$

$$2x - 3y + 4z = -1$$

**c**  $1 - x = y - 1 = 2z$  and  $x = 2 - t$

$$\frac{x-1}{-1} = \frac{y-1}{1} = \frac{z}{2} \quad y = 1 + 2t$$

$$z = t$$

(2, 1, 0) lies in the plane

$\begin{pmatrix} -1 \\ 1 \\ \frac{1}{2} \end{pmatrix}$  and  $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$  are vectors in the plane

$$\mathbf{n} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$$y - 2z = 1$$

**4 a**  $y = 0$       **b**  $z = 0$       **c**  $x = 0$

**5 a**  $\frac{1}{3} \times 4 \times h = 6 \therefore h = \frac{9}{2}$

0(0, 0, 0) A(2, 0, 0) B(2, 2, 0) C(0, 2, 0)  
V(0, 0,  $\frac{9}{2}$ )

**b**  $\vec{AB} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$   $\vec{AV} = \begin{pmatrix} -2 \\ 0 \\ \frac{9}{2} \end{pmatrix}$

$$\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 0 \\ \frac{9}{2} \end{pmatrix}$$

**c**  $\vec{CB} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$   $\vec{CV} = \begin{pmatrix} 0 \\ -2 \\ \frac{9}{2} \end{pmatrix}$

$$\mathbf{n} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -4 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ -18 \\ -8 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -4 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -4 \\ 9 \end{pmatrix}$$

$$-9y - 4z = -18$$

$$9y + 4z = 18$$

**d**  $\vec{VB} = \begin{pmatrix} 2 \\ 2 \\ -\frac{9}{2} \end{pmatrix}$   $\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 4 \\ -9 \end{pmatrix}$

**e**  $r = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 0 \\ 9 \end{pmatrix}$

$$x = 2 - 4\lambda, y = 0, z = 9\lambda$$

$$\lambda = \frac{x-2}{-4} \quad \lambda = \frac{z}{9}$$

$$\frac{x-2}{-4} = \frac{z}{9} \Rightarrow 9x - 36 = -4z$$

$$z = \frac{36-9x}{4}, y = 0$$

### Exercise 11N

**1**  $\mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \quad y = \sqrt{3}x - 3$

**Method 1**  $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 1 \\ \sqrt{3} \\ 0 \end{pmatrix}$

$$\cos \theta = \frac{|1-\sqrt{3}|}{\sqrt{2}\sqrt{4}} \therefore \theta = 75^\circ$$

**Method 2**  $m_1 = -1 \quad m_2 = \sqrt{3}$

$$\tan \mu = -1 \quad \tan \beta = \sqrt{3}$$

$$\therefore \theta = -45^\circ \therefore \beta = 60^\circ$$

$$\theta = 1\beta - \theta 1 = 105^\circ$$

$$\therefore \text{acute angle} = 75^\circ$$

**2**  $\frac{x-1}{2} = \frac{y}{3} = \frac{z}{1} \quad 2x = y = 3z$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{1}$$

$$\frac{x}{3} = \frac{y}{6} = \frac{z}{2}$$

$$\mathbf{u} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix} \quad \mathbf{u} \cdot \mathbf{v} = 6 + 18 + 2 = 26$$

$$|\mathbf{u}| = \sqrt{14} \quad |\mathbf{v}| = 7$$

$$\cos \theta = \frac{26}{7\sqrt{14}} \therefore \theta = 6.93^\circ$$

**3**  $\mathbf{r} = (2 - \theta) \mathbf{i} + (- + 2\theta) \mathbf{j} + (1 - \theta) \mathbf{k}$

$$\mathbf{r} = (2 + \beta) \mathbf{j} + (3 + \beta) \mathbf{k}$$

$$\mathbf{u} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \mathbf{u} \cdot \mathbf{v} = 1$$

$$\cos \theta = \frac{1}{\sqrt{6}\sqrt{2}} \therefore \theta = 73.2^\circ$$

**4** **a**  $\mathbf{u} = \begin{pmatrix} 1 \\ m_1 \\ m_1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 1 \\ m_2 \\ m_2 \end{pmatrix} \quad \mathbf{u} \cdot \mathbf{v} = 1 + m_1 m_2$

$$|\mathbf{u}| = \sqrt{1+m_1^2} \quad |\mathbf{v}| = \sqrt{1+m_2^2}$$

$$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{|\mathbf{u}| |\mathbf{v}|}$$

$$\therefore \cos \theta = \frac{|1+m_1 m_2|}{\sqrt{1+m_1^2} \sqrt{1+m_2^2}}$$

**b**  $\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} = \frac{m_1 - m_2}{1 + m_1 m_2}$

**c**  $\sec^2(\alpha - \beta) = 1 + \tan^2(\alpha - \beta)$

$$= 1 + \frac{(m_1 - m_2)^2}{(1 + m_1 m_2)^2}$$

$$= \frac{(1 + m_1 m_2)^2 + (m_1 - m_2)^2}{(1 + m_1 m_2)^2}$$

$$= \frac{1 + 2m_1 m_2 + m_1^2 m_2^2 + m_1^2 - 2m_1 m_2 + m_2^2}{(1 + m_1 m_2)}$$

$$= \frac{1 + m_1^2 + m_2^2 + m_1^2 m_2^2}{(1 + m_1 m_2)^2}$$

$$= \frac{(1 + m_1^2)(1 + m_2^2)}{(1 + m_1 m_2)}$$

$$\therefore \cos(\alpha - \beta) = \frac{|1 + m_1 m_2|}{\sqrt{1 + m_1^2} \sqrt{1 + m_2^2}} (> 0 \text{ since } \alpha > \beta)$$

**d**  $\cos \theta = \cos(\alpha - \beta)$  since  $\theta = \alpha - \beta$

### Exercise 11O

**1**  $\mathbf{r} = (1 - 2\lambda) \mathbf{i} + (1 - \lambda) \mathbf{j} + (-2 + \lambda) \mathbf{k} \quad \mathbf{u} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

$$2x - y + z = 5 \quad \mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\sin \theta = \frac{|\mathbf{u} \cdot \mathbf{n}|}{|\mathbf{u}| |\mathbf{n}|} = \frac{|-2|}{\sqrt{6}\sqrt{6}} \quad \theta = 195^\circ$$

**2**  $\frac{x-1}{3} = 2y = 3 - 2z \Rightarrow \frac{x-1}{3} = \frac{y}{1} = \frac{z-2}{-1}$

$$\Rightarrow \frac{x-1}{6} = \frac{y}{1} = \frac{z-2}{-1} \Rightarrow \mathbf{u} = \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix}$$

$$\mathbf{r} = (2 - 2\theta - 3\beta) \mathbf{i} + (1 - \theta + \beta) \mathbf{j} + (-2\theta + \beta) \mathbf{k}$$

$$\mathbf{n} = \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ -5 \end{pmatrix}$$

$$\mathbf{u} \cdot \mathbf{n} = 6 + 8 + 5 = 19$$

$$\sin \theta = \frac{19}{\sqrt{38}\sqrt{90}} \quad \therefore \theta = 19.0^\circ$$

**3**  $x - y + 3z = 1 \quad \mathbf{m} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

$$\mathbf{r} = (4 - 2\theta + 2\beta) \mathbf{i} + (1 - 3\beta) \mathbf{j} + (2 - \theta - \beta) \mathbf{k}$$

$$\mathbf{n} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \\ 6 \end{pmatrix} \quad \mathbf{m} \cdot \mathbf{n} = -3 + 4 + 18 = 19$$

$$\cos \theta = \frac{19}{\sqrt{11}\sqrt{61}} \quad \theta = 42.8^\circ$$

- 4 A(1, 0, 1) B(-1, 1, 0) C(2, 3, -1) D(-1, -1, -1)

a  $\vec{AB} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$   $\vec{AC} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

$$\mathbf{n} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix}$$

$$x - 5y - 7z = -6$$

b  $\vec{AD} = \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix}$   $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$$x = 1 + 2\lambda \quad y = \lambda \quad z = 1 + 2\lambda$$

$$\frac{x-1}{2} = y = \frac{z-1}{2}$$

c  $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$   $\mathbf{n} = \begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix}$   $\mathbf{u} \cdot \mathbf{n} = 2 - 5 - 14 = -17$

$$\sin \theta = \frac{17}{3\sqrt{75}} \quad \theta = 40.9^\circ$$

5  $\frac{x}{2} = ky = k - z \quad (2k - 1)x - ky + z = 5 + k$

$$\frac{x}{-2k} = \frac{y}{-1} = \frac{z-k}{k} \quad u = \begin{pmatrix} -2k \\ -1 \\ k \end{pmatrix} \quad n = \begin{pmatrix} 2k-1 \\ -k \\ 1 \end{pmatrix}$$

If the line and plane are parallel,  $\mathbf{u} \cdot \mathbf{n} = 0$

$$\text{so } -2k(2k-1) + k + k = 0$$

$$\Rightarrow -4k^2 + 4k = 0$$

$$\Rightarrow -4k(k-1) = 0 \Rightarrow k = 0 \text{ or } 1$$

### Exercise 11P

1  $\mathbf{r} = \mathbf{i} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

$$P(1 + \lambda, -2\lambda, 1 + \lambda) \quad x + y + 2z = 4$$

$$1 + \lambda - 2\lambda + 2(1 + \lambda) = 4$$

$$\lambda = 1 \quad P(2, -2, 2)$$

2  $\frac{x-1}{5} = \frac{y}{2} = \frac{z}{3} = \lambda \quad x - 1 + 5\lambda, y = 2\lambda, z = 3\lambda$

$$-x - y + 3z = 5$$

$$-1 - 5\lambda - 2\lambda + 9\lambda = 5$$

$$\therefore 2\lambda = 6 \quad \therefore \lambda = 3 \quad (16, 6, 9)$$

3  $x = 3k, y = 2 - 2k, z = 1 - k$

$$\mathbf{r} = (4 - 2\theta + \beta)\mathbf{i} + (1 - \beta)\mathbf{j} + (2 - \theta - 2\beta)\mathbf{k}$$

a  $x = 4 - 2\theta + \beta$

$$y = 1 - \beta$$

$$z = 2 - \theta - 2\beta$$

b  $3k = 4 - 2\theta + \beta \Rightarrow 3k + 2\theta - \beta = 4$

$$2 - 2k = 1 - \beta \Rightarrow -2k + \beta = -1$$

$$1 - k = 2 - \theta - 2\beta \Rightarrow k + \theta + 2\beta = 1$$

$$k = 0.6, \theta = 1.2, \beta = 0.2 \quad (1.8, 0.8, 0.4)$$

4  $\mathbf{r} = (1 + \lambda)\mathbf{i} + (1 + 2\lambda)\mathbf{j} + (1 + \lambda)\mathbf{k}, \mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$   
 $3x - y - z = 2 \quad n = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$

$\mathbf{u} \cdot \mathbf{n} = 3 - 2 - 1 = 0 \therefore \mathbf{u}$  and  $\mathbf{n}$  are perpendicular  
 $\therefore$  the line parallel to the plane (QED)

$(0, 0, -2)$  lies in the plane

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ lies in the plane}$$

$$\mathbf{r} = \lambda \mathbf{i} + 2\lambda \mathbf{j} + (-2 + \lambda) \mathbf{k} \text{ (other answers are possible)}$$

5  $P(1, 2, 3) \quad 2x + y - 5z = 1 \quad \mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$

line through p perpendicular to the plane:

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$$

point of intersection, I( $1 + 2\lambda, 2 + \lambda, 3 - 5\lambda$ )

$$2(1 + 2\lambda) + 2 + \lambda - 5(3 - 5\lambda) = 1$$

$$2 + 4\lambda + 2 + \lambda - 15 + 25\lambda = 1$$

$$30\lambda = 12 \Rightarrow \lambda = 0.4$$

$$I(1.8, 2.4, 1)$$

$$PI = \sqrt{(1.8-1)^2 + (2.4-2)^2 + (1-3)^2}$$

$$= \sqrt{4.8} = \frac{1}{10}\sqrt{480}$$

$$\text{distance} = \frac{2}{5}\sqrt{30} \text{ or } 2.19$$

### Exercise 11Q

1 a  $\frac{x}{2} = y - 1 = z \quad (1) \quad x = \frac{y+4}{3} = 3 - z \quad (2)$

$$x = 2y - 2 \quad 3x = y + 4$$

$$6y - 6 = y + 4$$

$$5y = 10 \Rightarrow y = 2, x = 2$$

sub in (1) sub in (2)

$$\frac{2}{2} = 2 - 1 = z \quad 2 = \frac{2+4}{3} = 3 - z$$

$$z = 1 \quad z = 1$$

$\therefore$  intersect at  $(2, 2, 1)$

b  $\mathbf{r} = (5 + 2\lambda)\mathbf{i} + (4 + \lambda)\mathbf{j} + (5 - 3\lambda)\mathbf{k}, x = y = z + 1$

$$P(5 + 2\lambda, 4 + \lambda, 5 - 3\lambda)$$

$$5 + 2\lambda = 4 + \lambda \text{ and } 4 + \lambda = 5 - 3\lambda + 1$$

$$\lambda = -1$$

$$4\lambda = 2$$

$$\lambda = \frac{1}{2}$$

Equations are inconsistent  $\therefore$  no point of intersection

2  $x - 3y + 8 = 2 \quad (1)$

$-x + y - 2z = 1 \quad (2)$

$(1) + (2) - 2y - z = 3 \Rightarrow z = -2y - 3$

$3(2) + (1) - 2x - 5z = 5 \Rightarrow z = \frac{-2x-5}{5}$

$\frac{-2x-5}{5} = -2y - 3 = z$

$\frac{2x+5}{5} = 2y + 3 = \frac{z}{-1}$

$\frac{x+2.5}{5} = \frac{y+1.5}{1} = \frac{z}{-2}$

3 a  $3x - y + z = 3 \quad \mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

$x = 3 - (2 - k) \lambda \quad y = (2k - 1) + \lambda$

$z = -1 + k\lambda \quad \mathbf{u} = \begin{pmatrix} k-2 \\ 1 \\ k \end{pmatrix}$

$\mathbf{n}$  and  $\mathbf{u}$  are collinear  $\therefore k = -1$

b  $3x - y + z = 3 \quad x = 3 - 3\lambda, y = -3 + \lambda,$

$z = -1 - \lambda$

$3(3 - 3\lambda) - (-3 + \lambda) + (-1 - \lambda) = 3$

$9 - 9\lambda + 3 - \lambda - 1 - \lambda = 3$

$-11\lambda = -8$

$\lambda = \frac{8}{11} \left( \frac{9}{11}, \frac{-25}{11}, \frac{-19}{11} \right)$

4 a  $L_1: y = 2x + 2, z = 3 - x$

$\frac{x}{1} = \frac{y-2}{2} = \frac{z-3}{-1}$  direction,  $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$L_2: \frac{x-1}{3} = \frac{y-1}{6} = \frac{1-z}{3} \Rightarrow \frac{x-1}{3} = \frac{y-1}{6} = \frac{z-1}{-3}$

direction,  $\mathbf{v} = \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix}$

$\mathbf{v} = 3\mathbf{u} \therefore L_1$  and  $L_2$  are parallel (QED)

b A(0, 2, 3) and B(1, 1, 1) are points in the plane

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

$5x - y + 3z = 7$

5  $x + y + 3z = 1 \quad x = 4 - y$  and  $z = -1$

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \frac{x}{1} = \frac{y-4}{-1} \quad \mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$\mathbf{u} \cdot \mathbf{n} = 1 - 1 + 0 = 0$

$\therefore \mathbf{u}$  and  $\mathbf{n}$  are perpendicular  $\therefore$  the line is parallel to the plane.

(0, 4, -1) lies on the line

$x + y + 3z = 0 + 4 + 3(-1) = 1$

$\therefore (0, 4, -1)$  also lies in the plane

$\therefore$  the plane contains the line (QED)

### Exercise 11R

1  $5x + y + 2z = 3 \quad (1) \quad (1) - (3) x - y = -2 \quad (4)$

$x + y + z = 3 \quad (2) \quad (1) - 2(2) 3x - y = -3 \quad (5)$

$4x + 2y + 2z = 5 \quad (3) \quad (5) - (4) 2x = -1$

$\therefore x = \frac{-1}{2} y = \frac{3}{2}$

From (2)  $z = 3 - x - y = 2 \left( \frac{-1}{2}, \frac{3}{2}, 2 \right)$

3  $x + y + z = 1 \quad (1) \quad (1) - (2) 2y = -2$

$x - y + z = 3 \quad (2) \quad y = -1$

$3x + y + 3z = 1 \quad (3)$

sub in (1) and (3)  $x + z = 2 \quad (4)$

$3x + 3z = 2 \quad (5)$

(4) and (5) are inconsistent  $\therefore$  no common point (QED)

4 a  $x + y + z = 0 \quad (1) \quad (2) - (1) ax - x = 0$

$ax + y + z = 0 \quad (2) \quad x(a-1) = 0$

$x + by + cz = z = 0 \quad (3) \quad x = 0 \text{ or } a = 1$

if  $x = 0, y + z = 0$  either  $y = z = 0$

$by + cz = 0 \text{ or } b = c$

If  $x = 0$ , either point of intersection  $(0, 0, 0)$  or intersect in a straight line.

if  $a = 1$  equations (1) and (2) are the same

$x + y + z = 0$

$x + by + cz = 0$

$x = y = z = 0$  or  $b = c = 1$  all 3 planes the same or  $b$  and  $c$  not both = 1 and intersect in a straight line.

$\therefore$  if  $a = 1$ , either a point of intersection  $(0, 0, 0)$  or intersect in a plane or interest in a line.

$\therefore$  the planes always have at least one point in common.

b For a straight line,  $b = c$  but  $a, b, c$  not all equal to 1 or  $a = 1$  and  $b$  and  $c$  both equal to 1

$x + 2y - 2z = 5 \pi_1$

$3x - 6y + 3z = 2 \pi_2$

$x - 2y + z = 7 \pi_3$

$\pi_2$  and  $\pi_3$  are parallel but not coincident,  $\pi$  intersects each of the other 2 please in a straight line but the 3 planes have no point in common.

**6 a**  $x + y + z = 2 \quad (1)$   $(1) + (2) 3x + 2z = 1 \quad (4)$   
 $2x - y + z = -1 \quad (2) \quad (1) + (3) 4x - 2z = 6 \quad (5)$   
 $3x - y - 3z = 4 \quad (3) \quad (4) + (5) 7x = 7$   
 $x = 1, z = -1, y = 2$   
 $(1, 2, -1)$

**b**  $x + y + z = 2 \quad (1) \quad (1) + (2) 3x + 2z = 1$   
 $2x - y + z = -1 \quad (2) \quad (1) + (3) 4x + (k+1)z = 6$   
 $3x - y + kz = 4 \quad (3)$   
For no common point,  $\frac{4}{3} = \frac{k+1}{2} (\neq 6)$   
 $8 = 3k + 3 \therefore k = \frac{5}{3}$

**7** Verify by diagrams

### Exercise 11S

**1** A  $x = 3 - t$  B  $x = 4 - 3t$   
 $y = 2t - 4 \quad y = 3 - 2t$

**a** A(3, -4) B(4, 3)

**b**  $V_A = \begin{pmatrix} -1 \\ 2 \end{pmatrix} V_B = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$

**c**  $\cos \theta = \frac{V_A \cdot V_B}{|V_A| |V_B|} = \frac{3-4}{\sqrt{5}\sqrt{13}} = \frac{-1}{\sqrt{65}} \quad \theta = 97.1^\circ$

**d**  $\vec{AB} = \begin{pmatrix} 4-3t \\ 3-2t \end{pmatrix} - \begin{pmatrix} 3-t \\ 2t-4 \end{pmatrix} = \begin{pmatrix} 1-2t \\ 7-4t \end{pmatrix}$

$AB = \sqrt{(1-2t)^2 + (7-4t)^2}$

This is a minimum when  $t = 1.5$  hours

**2 a**  $\vec{OP} = (5 + 10t)\mathbf{i} + (20 - 20t)\mathbf{j} + (30t - 10)\mathbf{k},$   
 $t > 0 \quad t = 0, P(5, 20, -10)$

**b** Cartesian equations:

$\frac{x-5}{10} = \frac{y-20}{-20} = \frac{z+10}{30} \quad (\times 10)$

$\frac{x-5}{1} = \frac{y-20}{-2} = \frac{z+10}{3}$

**c i**  $x + y + z = 55$

$5 + 10t + 20 - 20t + 30t - 10 = 55$

$20t = 40$

$t = 2$

**ii**  $(25, -20, 50)$

**iii**  $P_0 P_2 = \begin{pmatrix} 20 \\ -40 \\ 60 \end{pmatrix}$  distance

$= \sqrt{20^2 + (-40)^2 + 60^2} = 20\sqrt{14}$

**d**  $\vec{OQ} = \begin{pmatrix} 2t^2 \\ 1-2t \\ 1+t^2 \end{pmatrix} \quad t > 0$

**i**  $\vec{PQ} = \begin{pmatrix} 2t^2 - 5 - 10t \\ 1 - 2t - 20 + 20t \\ 1 + t^2 - 30t + 10 \end{pmatrix} = \begin{pmatrix} 2t^2 - 10t - 5 \\ 18t - 19 \\ t^2 - 30t + 11 \end{pmatrix}$

$PQ = \sqrt{(2t^2 - 10t - 5)^2 + (18t - 19)^2 + (t^2 - 30t + 11)^2}$

This is a minimum when  $t = 0.49598817$   
 $= 0.496 \text{ (3sf)}$

**ii**  $P(9.96, 10.1, 4.88)$   
 $Q(0.492, 0.00802, 1.25)$

**e i**  $a = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} b = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} c = \begin{pmatrix} 8 \\ -3 \\ 5 \end{pmatrix}$

$a - b = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} b - c = \begin{pmatrix} -6 \\ 2 \\ -3 \end{pmatrix}$

$a - b \neq k(b - c) \therefore a - b$  and  $b - c$  are non collinear (QED)

**ii** Q is not moving in a straight line

**3 a** B(1, 1, 0) C(0, 1, 0)  $\therefore \vec{OP} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$

C(0, 1, 0) G(0, 1, 1)  $\therefore \vec{OQ} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix}$

D(0, 0, 1) G(0, 1, 1)  $\therefore \vec{OR} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$

**b**  $\vec{PQ} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \vec{PR} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$

$n = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

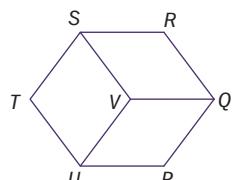
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$x + y + z = \frac{3}{2} \text{ or } 2x + 2y + 2z = 3$

c E(1, 0, 1) d(0, 0, 1)  $\therefore \vec{OS} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$

A(1, 0, 0) E(1, 0, 1)  $\therefore \vec{OT} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix}$

A(1, 0, 0) B(1, 1, 0)  $\therefore \vec{OU} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$

 S, T, U lie in the plane PQR

$$\vec{PQ} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}, \vec{QR} = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \vec{RS} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}, \vec{ST} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}$$

$$\vec{TU} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \vec{UP} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

all sides are of length  $\frac{1}{\sqrt{2}}$

$\cos P = \cos Q = \cos R = \cos S = \cos T$

$$= \cos U = \frac{\frac{-1}{4}}{\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}} = \frac{-1}{2}$$

$\therefore$  all angles are  $120^\circ$   $\therefore$  PQRSTU is a regular hexagon

Area parallelogram PQVU =  $|\vec{PQ} \times \vec{PU}|$

$$\begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \times \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} \therefore \text{area PQVU} = \frac{\sqrt{3}}{4}$$

$$\therefore \text{area hexagon} = \frac{3\sqrt{3}}{4}$$

d  $\vec{OF} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = n$   $\therefore \vec{OF}$  is perpendicular to the plane PQR (QED)

e General point on OF is  $(\lambda, \lambda, \lambda)$

$$2x + 2y + 2z = 3 \quad 2\lambda + 2\lambda + 2\lambda = 3$$

$$\therefore \lambda = \frac{3}{6} = \frac{1}{2} \quad I\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

f  $\vec{IF} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \therefore \text{distance IF} = \frac{\sqrt{3}}{2}$



### Review exercise

1  $\mathbf{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$  a  $3\mathbf{u} - 2\mathbf{v} = \begin{pmatrix} 9 \\ 12 \end{pmatrix} - \begin{pmatrix} -2 \\ 10 \end{pmatrix} = \begin{pmatrix} 11 \\ 2 \end{pmatrix}$

b  $|\mathbf{u}| = 5, |\mathbf{v}| = \sqrt{26}$   $\mathbf{u} + \mathbf{v} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$   
 $\therefore |\mathbf{u} + \mathbf{v}| = \sqrt{85}$

2  $\mathbf{u} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{a} = \mathbf{i} + \mathbf{j}, \mathbf{b} = \mathbf{i} + \mathbf{k}$

$\mathbf{c} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$

$\mathbf{u} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$

$$\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

$$\therefore \alpha + \beta + 2\gamma = 3 \quad (1) \quad (1) - (2) \quad \beta + 3\gamma = 4 \quad (4)$$

$$\alpha - \gamma = -1 \quad (2) \quad \beta - \gamma = 1 \quad (3)$$

$$\beta - \gamma = 1 \quad (3) \quad (4) - (3) \quad 4\gamma = 3$$

$$\gamma = \frac{3}{4}$$

$$\gamma = \frac{3}{4}, \beta = \frac{7}{4}, \alpha = -\frac{1}{4} \therefore \mathbf{u} = -\frac{1}{4}\mathbf{a} + \frac{7}{4}\mathbf{b} + \frac{3}{4}\mathbf{c}$$

3 a  $\mathbf{a} = \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}, |\mathbf{a}| = 5\sqrt{2} \therefore \text{unit vector} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{-4}{5\sqrt{2}} \\ \frac{3}{5\sqrt{2}} \end{pmatrix}$

b  $\pm \frac{5}{5\sqrt{2}} \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix} = \pm \frac{-5}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{pmatrix}$

4  $\mathbf{u} = \cos \alpha \cos \beta \mathbf{i} + \sin^2 \alpha \cos^2 \beta \mathbf{j} + \sin^2 \beta \mathbf{k}$

$$|\mathbf{u}|^2 = \cos^2 \alpha \cos^2 \beta + \sin^2 \alpha \cos^2 \beta + \sin^2 \beta$$

$$= \cos^2 \beta (\cos^2 \alpha + \sin^2 \alpha) + \sin^2 \beta$$

$$= \cos^2 \beta + \sin^2 \beta$$

$$= 1$$

$$\therefore |\mathbf{u}| = 1 \therefore \mathbf{u} \text{ is a unit vector}$$

5  $\mathbf{u} = \mathbf{i} + \tan \alpha \mathbf{j}, \mathbf{v} = \tan \beta \mathbf{i} + \mathbf{j}$

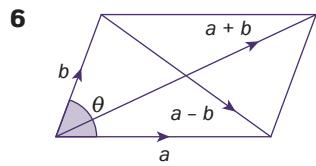
$$\cos r = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} = \frac{\tan \beta + \tan \alpha}{\sqrt{1 + \tan^2 \alpha} \sqrt{1 + \tan^2 \beta}}$$

$$= \frac{\tan \beta + \tan \alpha}{\sec \alpha \sec \beta}$$

$$= (\tan \beta + \tan \alpha) \cos \alpha \cos \beta$$

$$= \sin \beta \cos \alpha + \sin \alpha \cos \beta$$

$$\begin{aligned}\cos \gamma &= \sin(\alpha + \beta) \\ &= \cos\left(\frac{\pi}{2} - (\alpha + \beta)\right) \\ \therefore \gamma &= \frac{\pi}{2} - (\alpha + \beta) \therefore \alpha + \beta + \gamma = \frac{\pi}{2}\end{aligned}$$



- a**  $|\mathbf{a}| = 1$   $|\mathbf{b}| = 1$   
 $|\mathbf{a} - \mathbf{b}|^2 = 1^2 + 1^2 - 2(1)(1) \cos \theta$   
 $|\mathbf{a} - \mathbf{b}|^2 = 2 - 2 \cos \theta$   
 $|\mathbf{a} + \mathbf{b}|^2 = 1^2 + 1^2 - 2(1)(1) \cos(\pi - \theta)$   
 $= 2 - 2 \cos(\pi - \theta)$   
 $|\mathbf{a} + \mathbf{b}|^2 = 2 + 2 \cos \theta$
- b**  $|\mathbf{a} + \mathbf{b}| = 2|\mathbf{a} - \mathbf{b}| \Rightarrow |\mathbf{a} + \mathbf{b}|^2 = 4|\mathbf{a} - \mathbf{b}|^2$   
 $2 + 2 \cos \theta = 4(2 - 2 \cos \theta)$   
 $2 + 2 \cos \theta = 8 - 8 \cos \theta$   
 $10 \cos \theta = 6 \therefore \cos \theta = \frac{3}{5} (0 \leq \theta \leq \frac{\pi}{2})$   
 $\sin^2 \theta = 1 - \cos^2 \theta = 1 - \frac{9}{25} = \frac{16}{25} \therefore \sin \theta = \frac{4}{5}$

**7 a**  $\mathbf{r} = (2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) + \alpha(2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) + \beta(\mathbf{j} - 2\mathbf{k})$   
 $x = 2 + 2\alpha, y = 3 - 3\alpha + \beta, z = 4 + 2\alpha - 2\beta$

**b**  $\mathbf{r} = (\mathbf{i} - 2\mathbf{k}) + \alpha(-2\mathbf{i} + \mathbf{k}) + \beta(-\mathbf{j})$   
 $x = 1 - 2\alpha, y = -\beta, z = -2 + \alpha$

**8**  $5 + 3\lambda = -2 + 4\mu \quad (1)$

$1 - 2\lambda = 2 + \mu \quad (2)$

From (2)  $\mu = -2\lambda - 1$

$\text{Sub in (1)} \quad 5 + 3\lambda = -2 - 8\lambda - 4$

$11\lambda = -11 \therefore \lambda = -1 \quad \alpha = 1 \quad P(2, 3)$



## Review exercise

**1**  $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$   $\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$   $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|}$   
 $\cos \theta = \frac{2+3}{\sqrt{11}\sqrt{5}} = \frac{5}{\sqrt{55}}$   
 $\theta = 48^\circ \text{ (nearest degree)}$

**2 a**  $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$   $\overrightarrow{AC} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$   
 $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$   
 $x = 1 - 2\alpha - \beta, y = 1, z = 2\alpha - \beta$

$\mathbf{n} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix}$$

$-4y = -4 \text{ or } y = 1$

**b**  $\overrightarrow{AB} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$   $\overrightarrow{AC} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

$$\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

$x = -1 - \alpha + \beta, y = 1 - 2\alpha, z = 1 + \alpha - 2\beta$

$$\mathbf{n} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$$

$4x - y + 2z = -3$

**3 a**  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

or  $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = -9$

**b**  $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$   $\mathbf{r}(\mathbf{i} - \mathbf{j}) = 1$

**4 a**  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$   
 $2x - 3y + 4z = 29$

**b**  $\overrightarrow{AB} = \begin{pmatrix} -6 \\ 0 \\ -3 \end{pmatrix}$   $\overrightarrow{AC} = \begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix}$   $\mathbf{n} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 4 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -1 \\ 4 \end{pmatrix}$$

$-2x - y + 4z = -12$

$2x + y - 4z = 12$

**c**  $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$   $\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$2x - y = 4$$

**d**  $\vec{AB} = \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}$   $\mathbf{n} = \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$$

$$5y + 2z = -11$$

**e**  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$x = 3$$

**f**  $\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$x - y - z = 0$$

**g**  $x + y + 5z = 0 \quad (1)$

$$2x + 3y + 12z = 0 \quad (2)$$

$$2 - 2(1) y + 2z = 0$$

$$\text{Let } z = \lambda, y = -2\lambda, x = -y - 5z = 2\lambda - 5\lambda$$

$$x = -3\lambda$$

Line of intersection is  $\mathbf{r} = \lambda \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$

$$\mathbf{n} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 0$$

$$-x + 2y + z = 0$$

$$\text{or } x - 2y - z = 0$$

**5**  $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$   $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{|\mathbf{u}| |\mathbf{v}|} = \frac{|2-1-2|}{\sqrt{6}\sqrt{6}} = \frac{1}{6}$$

$$\theta = 80.4^\circ$$

**6**  $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

$$\mathbf{r} = \mathbf{i} + 4\mathbf{j} + 5\mathbf{k} + \alpha(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$$

$$1 + \lambda = 1 + 2\alpha \quad (1)$$

$$1 + 2\lambda = 4 + \alpha \quad (2)$$

$$1 + 3\lambda = 5 + 2\alpha \quad (3)$$

From (1)  $\lambda = 2\alpha$

Sub in (2)  $1 + 4\mu = 4 + \mu$

$$\therefore 3\alpha = 3 \therefore \alpha = 1, \lambda = 2$$

Check in (3)  $1 + 3(2) = 5 + 2(1)$

$$7 = 7$$

$\therefore L_1$  and  $L_2$  are concurrent

point of intersection is  $(3, 5, 7)$

$$\mathbf{r} = 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k} + \alpha(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \beta(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$$

**7**  $x + y + z = 3 \quad (1)$   $(2) - (1) \quad x - 3z = -3 \quad (4)$

$$2x + y - 2z = 0 \quad (2)$$
  $(3) + 2(1) \quad 5x + 7z = 29 \quad (5)$

$$3x - 2y + 5z = 23 \quad (3)$$

$$(5) - 5(4) \quad 22z = 44$$

$$\therefore z = 2, x = 3$$

$$y = 2z - 2x = -2$$

$$\therefore x = 3, y = -2, z = 2$$

$(3, -2, 2)$  is the point of intersection of the 3 planes represented by the equations.

**8 a**  $3x + y + z = 1 \quad (1)$   $(1) - (2) \quad 2x + 2z = -3 \quad (4)$

$$x + y - z = 4 \quad (2)$$
  $(1) - (3)$

$$x + (1 - b)z = 1 - a \quad (5)$$

$$2x + y + bz = a \quad (3)$$

$$(4) - 2(5) \quad 2z - 2(1 - b)z = -3 - 2(1 - a)$$

$$2bz = -5 + 2a$$

$$z = \frac{2a - 5}{2b}$$

$$2x = -3 - 2z = -3 - \frac{(2a - 5)}{b} = \frac{-3b - 2a + 5}{b}$$

$$x = \frac{5 - 2a - 3b}{2b}$$

$$y = 4 - x + z = \frac{8b - (5 - 2a - 3b) + 2a - 5}{2b}$$

$$y = \frac{-10 + 4a + 11b}{2b}$$

$$x = \frac{5 - 2a - 3b}{2b}, y = \frac{4a + 11b - 10}{2b}, z = \frac{2a - 5}{2b}$$

**b**  $b = 0$  equations (4) and (5) become

$$2x + 2z = -3$$

$$x + z = 1 - a$$

For a non-unique solution,  $1 - a = -\frac{3}{2}$

$$\therefore a = \frac{5}{2}$$

$a = \frac{5}{2}, b = 0$  the planes represented by the 3 equations intersect in a line.